Capital-Labor Substitution, Structural Change and Growth

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Abstract

There is growing interest in multi-sector models that combine aggregate balanced growth, consistent with the well-known Kaldor facts, with systematic changes in the sectoral allocation of resources, consistent with the Kuznets facts. Although variations in the income elasticity of demand across goods played an important role in initial approaches, recent models stress the role of supply-side factors in this process of structural change, in particular sector-specific technical change and sectoral differences in factor proportions. We explore a general framework that features an additional supply-side mechanism and also encompasses these two known mechanisms. Our model shows that sectoral differences in the degree of capital-labor substitutability – a new mechanism – are a driving force for structural change. When the flexibility to combine capital and labor differs across sectors, a factor rebalancing effect is operative. It tends to make production in the more flexible sector more intensive in the input that becomes more abundant. As a result, growth rates of sectoral capital-labor ratios can differ and, if this effect dominates, shares of each factor used in a given sector can move in different directions. We identify conditions under which this occurs and analyze the dynamics of such a case. We also provide some suggestive evidence consistent with this new mechanism. A quantitative analysis suggests that this channel was an important contributor to structural change out of agriculture in the United States.

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1 Introduction

The theoretical literature on economic growth has traditionally been interested in models that exhibit a balanced growth path, a trajectory where the growth rate of output, the capital-output ratio, the return to capital, and factor income shares are constant. It has become standard to impose restrictions on preferences and technology to be consistent with these "Kaldor facts" (Kaldor, 1963). Nonetheless, underlying this balanced process at the aggregate level, there are systematic changes in the composition of output at a more disaggregated level – a secular process of structural change. The seminal work of Clark (1940) and Kuznets (1966) already documented a facet of this structural transformation, particularly the continuous decrease in the share of agriculture in output and employment that accompanies long-run increases in income per capita. More recently, several authors (see for instance Kongsamut et al. 2001; Buera and Kaboski, 2012) have drawn attention to the increasing importance of the service sector. Kongsamut et al. (2001) have dubbed this second set of empirical regularities associated with the process of structural change the "Kuznets facts".

In recent years, several multi-sector growth models that address both the Kaldor and the Kuznets facts have been proposed. Inspired by the early contributions of Baumol (1967) and Matsuyama (1992), this literature has identified several channels that can drive structural change and are still consistent with a balanced growth path. One can classify these channels into two categories: preference-driven and technology-driven structural change.[1] In the first category (see for instance Kongsamut et al. (2001), Foellmi and Zweimüller (2008) or Boppart (2014)), structural change is driven by differences in the income elasticity of demand across goods.[2] As capital accumulates and income rises, these differences shift demand, and therefore resources and production, from goods with low demand elasticity, such as food or necessities, to high demand elasticity goods, such as services or luxuries. In the second category, where technological differences across sectors play the dominant role, two alternative mechanisms have been proposed. The first mechanism, recently formalized by Ngai and Pissarides (2007), hereafter NP, works through differences in the sectoral rates of TFP growth. The second mechanism, explored by Acemoglu and Guerrieri (2008), hereafter

[1]Most papers in this literature use a closed economy framework. In this context, the interaction between preferences and technology determines sectoral allocations. In an open economy, given world prices, sectoral allocations are only determined by the supply side of the model. See Matsuyama (1992) and Ventura (1997) for models of structural change in an open economy and Alvarez-Cuadrado and Poschke (2011) for the relevance of the closed-economy assumption in the context of structural change out of agriculture.

[2]There is a large body of work that assumes non-homotheticity as a source for structural change. See also Echeverria (1997), Laitner (2000), Caselli and Coleman (2001), Gollin, Parente and Rogerson (2007), and Restuccia and Duarte (2010). Boppart (2014) considers non-homothetic preferences and differential TFP growth jointly.

AG, places its emphasis on sectoral differences in factor proportions, i.e. differences in the elasticity of output with respect to capital across sectors. As sectoral levels of TFP diverge or capital accumulates, these two channels generate a process of structural change.[3]

The main objective of this paper is to explore an additional source for technology-driven structural change consistent with quasi-balanced growth at the aggregate level: sectoral differences in the elasticity of substitution between capital and labor. Intuitively, if the degree of flexibility with which capital and labor can be combined to produce output varies across sectors, changes in the wage to rental rate ratio that accompany aggregate growth lead to systematic reallocations of factors of production and to changes in the sectoral composition of output. In this paper, we formalize this simple intuition. In our model, as the aggregate capital-labor ratio and the wage to rental rate ratio increase, the sector with higher elasticity of factor substitution – the more flexible sector – is in a better position to substitute away from the progressively relatively more expensive input, labor, and into the progressively cheaper one, capital, compared to the less flexible sector. As a result, differences in the sectoral elasticity of substitution between capital and labor induce a process of structural change.

Our exercise is motivated by three observations. Firstly, there is direct econometric evidence that the elasticity of substitution differs across sectors. Most recently, Herrendorf, Herrington and Valentinyi (2015) have estimated sectoral CES production functions using postwar U.S. data. Their results indicate not only substantial deviations from the Cobb-Douglas benchmark, but also pronounced differences across sectors, with an elasticity of substitution between capital and labor below one for both services and manufacturing, and an elasticity well above one for agriculture. Secondly, there is evidence that capital-labor ratios grow at different rates in different sectors. Most notably, the yearly average growth rate of the capital-labor ratio in U.S. agriculture has been 1.9% since 1960, compared to 1.4% for the economy as a whole.[4] As we will see, under free factor mobility, this cannot arise if sectoral production functions are Cobb-Douglas, and thus also points towards differences in substitution elasticities across sectors. Thirdly, factor income shares have evolved differently across sectors. For instance, Zuleta and Young (2013) construct sectoral labor income shares using data from the U.S. 35 sector KLEM database from 1960 to 2005 (Jorgenson, 2007). Over this period, the labor income share in agriculture fell by 15 percentage points, roughly

[3]Baumol (1967) suggests several mechanisms behind structural change, specifically "innovations, capital accumulation, and economies of large scale" (p. 415). Recently, Buera and Kaboski (2012) developed a model where structural change results from differences in the scale of productive units across sectors. This is yet another source of technology-driven structural change.

[4]Data from Herrendorf, Herrington and Valentinyi (2015).

three times its change in the rest of the economy.[5] While factor income shares have a whole set of determinants, it is clear that this observation is inconsistent with Cobb-Douglas production functions at the sectoral level, and also points towards sectoral differences in factor substitutability.

These changes in sectoral capital-labor ratios and factor income shares coincided with substantial structural change. For instance, the contribution of agricultural value added to U.S. GDP declined by 90% from the end of World War II to 2010, from 9% to just 1% of GDP. Over the same period, hours worked of persons engaged in agriculture declined by a similar proportion, from 18% to barely 2% of total hours worked. In contrast to this, the fraction of capital used in agriculture only declined by half, from 11% to 5%, illustrating again the much faster growth in the capital-labor ratio in agriculture. The crucial role played by the elasticity of substitution in the evolution of the capital-labor ratios and the factor income shares suggests that this elasticity may also play an important role in the process of structural change.[6]

We thus analyze theoretically how differences in factor substitutability affect structural change. The framework we use for our analysis is a two-sector version of the Solow model. Final output is produced using a CES aggregator that combines two intermediate inputs. These are in turn sectoral outputs produced under two different CES production functions, using capital and labor. By varying parameter restrictions, this simple framework allows us not only to analyze the new mechanism that we are proposing, but also to capture the essence of the two supply-side mechanisms stressed in the previous literature.

To begin, we identify conditions on parameters that determine how factor allocations react to capital accumulation and technical change. These conditions arise from the balance of three effects: a *relative price effect*, a *relative marginal product effect*, and *a factor rebalancing effect*. While the former two are already present in the previous literature on technology-driven structural change, the latter is new and arises only in the presence of dif-

[5]Aside from the high frequency variations in factor income shares, documented for instance by Blanchard (1997), Caballero and Hammour (1998) and Bentolila and Saint-Paul (2003), there has been a recent reappraisal of the long-run constancy of the aggregate labor income share. Two recent contributions, Elsby, Hobijn, and Sahin (2013) and Karabarbounis and Neiman (2014), have documented a secular decline in the labor income share in the United States and at a global level. In a companion paper, Alvarez-Cuadrado, Long, and Poschke (2015), we explore the implications of sectoral differences in the elasticity of substitution between capital and labor for this decline in the aggregate labor income share.

[6]Recall that this elasticity was first introduced by Hicks in his seminal work, *The Theory of Wages* (1932), to explore the distribution of income between factors in a growing economy. First, Pitchford (1960) and, recently, Klump and de La Granville (2000) and Jensen (2003) analyze the relationship between the CES production function and the possibility of permanent growth in a neoclassical model of capital accumulation. The latter explores multi-sector models, but differs in focus from our work.

ferences in the elasticity of substitution across sectors. It reflects that, as a factor becomes more abundant and therefore relatively cheaper, e.g. due to capital accumulation, the more flexible sector increases its use of that factor more. We show that if this effect is strong enough, it is possible for sectoral capital and labor allocations to move in opposite directions with capital accumulation or technical change. More generally, its presence leads to differences in the evolution of capital-labor ratios. The evolution of value added shares of the two sectors depends on whether an increase in the capital-labor ratio makes the economy's endowments of capital and labor more or less "balanced." To build intuition, we also present results for some special parametric cases and show some illustrative simulations.

In addition to static allocations, we can characterize the dynamics of the model in a tractable special case that illustrates the effects of differences in the substitution elasticity. Concretely, we assume that final output is produced under a Cobb-Douglas technology and that the sectoral technologies are identical except for the sectoral elasticity of substitution between capital and labor. In this case, the factor rebalancing effect dominates. Then, as the aggregate capital-labor ratio increases, the fractions of capital and labor allocated to the flexible sector move in opposite directions: the flexible sector absorbs more capital and releases labor. Intuitively, as capital accumulates and the wage to rental rate ratio increases, the flexible sector will tend to substitute from the now more expensive input, labor, towards the relatively cheaper one, capital, at a higher rate than the less flexible sector is able to do. Sectoral capital-labor ratios and factor income shares thus can evolve differently in the two sectors. In the context of this simple framework, we show that the economy converges to a balanced growth path consistent with the Kaldor facts, with structural change taking place along the transition only. These results, derived under a constant saving rate, readily extend to an endogenous saving environment.

It is worth stressing that the differences in the degree of factor substitution are a distinct driver of structural change compared to the differences in factor proportions stressed by AG. This is true both in terms of implications, as just discussed, and conceptually. Conceptually, for given factor prices, factor proportions are determined by the interaction between the elasticity of substitution and the distributive parameter (the α in the Cobb-Douglas technology). AG focus on sectoral differences in this latter parameter, while our model stresses the effects of differences in the former.

Finally, the mechanism proposed here also differs from those in AG and NP in that structural change is driven by changes in the relative price of factors, not the relative price of sectoral outputs. As is well known, the factor allocation changes with growth in response to changes in the relative price of the two intermediate goods in both the AG and NP models. Instead, with differences in the elasticity of substitution, the factor allocation changes in response to changes in the ratio of the wage rate to the rental rate of capital.

We round off the paper with a brief quantitative analysis of structural change out of agriculture. To begin, we show some suggestive evidence from cross-country data, which clearly reveal that richer economies not only allocate fewer resources to agriculture, but also have much more capital intensive agricultural sectors. This is consistent with a higher elasticity of substitution between capital and labor in agriculture. We then show that a version of our model calibrated using estimates of sectoral substitution elasticities from Herrendorf, Herrington and Valentinyi (2015) fits structural change out of agriculture in the U.S. very well. Notably, it renders the different paths of capital and labor allocated to agriculture very closely. A decomposition analysis also confirms the importance of differences in substitution elasticities for this episode of structural change.

The mechanism illustrated in this paper is also related to that in Ventura (1997) and to the literature on capital-skill complementarity initiated by Krusell, Ohanian, Rios-Rull and Violante (2000). Ventura (1997) presents a multi-country growth model where final output is produced combining two intermediate goods, one of which is produced using only capital, while the other uses only labor. There is free trade in both intermediate goods, although international factor movements are not permitted. In this context, as a country accumulates capital, resources are moved from labor-intensive to capital-intensive uses – a process of structural change – while international trade converts this excess production of capital-intensive goods into labor-intensive ones. Krusell et al. (2000) present a neoclassical growth model where the elasticity of substitution between capital equipment and unskilled labor is higher than that between capital equipment and skilled labor. As a result, as capital accumulates, the wage for skilled workers increases relative to that of unskilled workers, in line with the increase in the skill premium observed over the last 20 years of the past century. Finally, the results we obtain can also be useful for the analysis of structural change in terms of other factors of production for which substitutability may differ across sectors. For instance, Reshef (2013), Buera, Kaboski and Rogerson (2015) and Wingender (2015) suggest that this could be the case for skilled and unskilled labor.

The paper is organized as follows. Section 2 sets out the basic model. Section 3 analyzes optimal static allocations, and Section 4 analyzes growth paths for a special case with unequal sectoral capital-labor substitution. Section 5 provides some suggestive cross-country evidence and a quantitative analysis of structural change out of agriculture in the U.S..

The conclusions are summarized in Section 6, while the Appendix includes proofs and some technical details.

2 A general model of structural change

We model a closed economy where a single final good is produced under perfect competition by combining the output of two intermediate-good sectors, Y_s, where $s = 1, 2$, according to a CES technology with elasticity of substitution $\varepsilon \in [0, \infty)$:

$$Y(t) = F(Y_1(t), Y_2(t)) = \left[\gamma Y_1(t)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma) Y_2(t)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{1}$$

where $\gamma \in (0, 1)$ is the distributive share.[7] Both intermediate-good sectors use two inputs, labor, L, and capital, K. The labor force grows at a rate n and capital depreciates at a rate δ. The aggregate resource constraint requires the sum of consumption, C, and gross investment, I, to be equal to output of the final good

$$\dot{K}(t) + \delta K(t) + C(t) \equiv I(t) + C(t) = Y(t) \tag{2}$$

where the dot denotes the change in a variable. Under the assumption that a fixed fraction of output, v, is saved and invested every period, equation (2) yields the following law of motion for the capital stock,

$$\dot{K}(t) = vY(t) - \delta K(t). \tag{3}$$

The two intermediate goods are produced competitively according to

$$Y_s(t) = \left[(1-\alpha_s)(M_s(t) L_s(t))^{\frac{\sigma_s-1}{\sigma_s}} + \alpha_s K_s(t)^{\frac{\sigma_s-1}{\sigma_s}}\right]^{\frac{\sigma_s}{\sigma_s-1}} \tag{4}$$

where $\alpha_s \in (0, 1), \sigma_s \in [0, \infty), M_s, L_s$, and K_s are respectively the distributive share, the elasticity of substitution, and the levels of technology, employment, and capital for sector s.[8] Both inputs are fully utilized:

$$\begin{aligned} L_1(t) + L_2(t) &= L(t), \\ K_1(t) + K_2(t) &= K(t). \end{aligned}$$

Moreover, given the assumption of finite ε, σ_1 and σ_2, both sectors use strictly positive quantities of both inputs.

[7] When $\varepsilon = 1$, this equation becomes $Y = Y_1^{\gamma} Y_2^{1-\gamma}$.

[8] Again, the possibility that $\sigma_s = 1$ is admitted, for one or for both sectors. Also note that the capital income share in sector s is not simply equal to α_s unless $\sigma_s = 1$.

Technological progress in each sector is exogenous and exhibits a constant growth rate:

$$\frac{\dot{M}_s(t)}{M_s(t)} = m_s \geqslant 0, \quad s = 1, 2. \tag{5}$$

This general setup allows for a novel mechanism, where differences in the degree of capital-labor substitutability across sectors drive the evolution of sectoral factor allocations. At the same time, the setup includes as special cases several of the mechanisms described in the theoretical literature on structural change. That is, setting $\varepsilon \neq 1$, $\sigma_1 = \sigma_2 = 1$, $m_2 = m_1$ and $\alpha_1 \neq \alpha_2$ yields a model of structural change (and non-balanced growth) driven by differences in factor proportions and capital deepening, as in AG. Instead, setting $\varepsilon \neq 1$, $\sigma_1 = \sigma_2 = 1$, $\alpha_1 = \alpha_2$ and $m_2 \neq m_1$ leads to a model where structural change is driven by differences in the growth rates of sectoral TFP, as in NP.

We break the solution of our problem into two steps. First, given the vector of state variables at any point in time, (K, L, M_1, M_2), the allocation of factors across sectors is chosen to maximize final output, (1). This is the static problem, solved in the next section. Second, given factor allocations at each date, the time path of the capital stock follows the law of motion (3). The dynamic problem consists of examining the stability of this process. This is analyzed in Section 4.

3 The static problem

Let us denote the rental rate, the wage rate, the prices of the intermediate goods and the price of the final good by $R \equiv r + \delta$, w, p_1, p_2 and P respectively.[9] It will prove useful to define capital per worker, $k = K/L$, and the shares of capital and labor allocated to sector 1 as

$$\kappa \equiv \frac{K_1}{K} \quad \text{and} \quad \lambda \equiv \frac{L_1}{L}. \tag{6}$$

Throughout, we will also assume without loss of generality that $\sigma_2 > \sigma_1$, i.e. sector 2 is the more flexible sector.

The optimal allocation involves two trade-offs: the optimal allocation of resources across sectors, and the optimal balance of resources, or the optimal capital-labor ratio, in each sector. The former is summarized by the *labor mobility condition (LM)* and the latter by the *contract curve (CC),* which we derive and characterize next.

[9] We drop the time indicators when there is no risk of ambiguity.

3.1 The contract curve (CC)

At any point in time and for any prices, free mobility of capital and labor across sectors implies the equalization of the marginal value products of these factors across sectors:

$$p_1\alpha_1\left(\frac{Y_1}{K_1}\right)^{\frac{1}{\sigma_1}} = p_2\alpha_2\left(\frac{Y_2}{K_2}\right)^{\frac{1}{\sigma_2}} = R \tag{7}$$

$$p_1(1-\alpha_1)\left(\frac{Y_1}{L_1}\right)^{\frac{1}{\sigma_1}} M_1^{\frac{\sigma_1-1}{\sigma_1}} = p_2(1-\alpha_2)\left(\frac{Y_2}{L_2}\right)^{\frac{1}{\sigma_2}} M_2^{\frac{\sigma_2-1}{\sigma_2}} = w \tag{8}$$

Combining these equations yields the contract curve:

$$CC(\kappa,\lambda,k,M_1,M_2) \equiv \frac{1-\alpha_1}{1-\alpha_2}\frac{\alpha_2}{\alpha_1}\frac{M_1^{\frac{\sigma_1-1}{\sigma_1}}}{M_2^{\frac{\sigma_2-1}{\sigma_2}}}k^{\frac{1}{\sigma_1}-\frac{1}{\sigma_2}}\frac{\kappa^{\frac{1}{\sigma_1}}}{(1-\kappa)^{\frac{1}{\sigma_2}}}\frac{\lambda^{-\frac{1}{\sigma_1}}}{(1-\lambda)^{-\frac{1}{\sigma_2}}} = 1. \tag{CC}$$

This curve describes the set of points where the marginal rates of technical substitution are equalized across sectors. In a slight abuse of notation, we will refer to its left hand side as (CC).

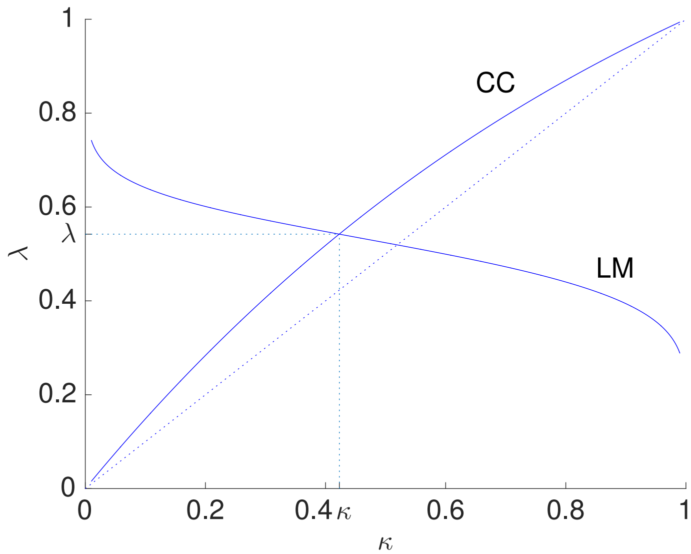
For the analysis of the optimal allocation of capital and labor across sectors and its shifts with development, it is useful to depict the contract curve and the labor mobility condition in κ,λ-space (see Figure 1). Since (CC) is increasing in κ and decreasing in λ, the CC curve is upward-sloping in this space. It is also clear from equation (CC) that it connects the origin with the point (1,1). (To see this, bring the fraction involving λ to the right hand side.)

The positive slope of the curve reflects the complementarity of capital and labor in production: Increasing the amount of capital allocated to sector 1 increases the marginal product of labor in sector 1 relative to sector 2, and therefore also calls for allocating more labor to sector 1. Note that, of course, the curve can also be transformed to show the optimal relationship between capital-labor ratios in the two sectors.

A few special cases are worth of interest. In the case where both sectors have Cobb-Douglas production functions (as in AG and NP), $\sigma_1 = \sigma_2 = 1$, (CC) implies that $\lambda/(1-\lambda)$ is a linear function of $\kappa/(1-\kappa)$, and that k_1 and k_2 are proportional. In this case, (CC) reacts neither to changes in k nor to changes in M_1 or M_2. Relative capital intensity in this scenario is governed by α_1 and α_2. When $\alpha_1 > \alpha_2$, sector 1 is more capital-intensive, and therefore $\kappa > \lambda$ for all levels of k and technology. When $\alpha_1 = \alpha_2$, $\kappa = \lambda$, and capital-labor ratios are always equated across sectors.

When σ is common but not equal to one, $\kappa/(1-\kappa)$ and $\lambda/(1-\lambda)$ are also proportional and unaffected by changes in k or by proportional changes in M_s in both sectors. They do

Figure 1: The contract curve (CC) and the labor mobility condition (LM)



Notes: Parameter values: $\gamma = 0.5$, $\varepsilon = 0.5$, $\sigma_1 = 1$, $\sigma_2 = 1.2$, $\alpha_1 = 0.3$, $\alpha_2 = 0.4$, $M_1 = M_2 = 1$, $k = 1$. The dotted line is the 45-degree line.

however respond to changes in relative M_s. Capital intensity is higher in the sector with larger α_s or with larger (smaller) M_s, if $\sigma < (>)1$.

It is clear from this discussion that in the Cobb-Douglas case, and to some extent even more generally when sectoral substitution elasticities are identical, the contract curve is not very interesting. In these cases, the location of the CC curve only influences relative factor intensity. All "action" in terms of structural change, however, is driven by movements in the labor mobility curve. This is not true anymore when σ differs across sectors. In this case, differences in σ determine how the curve shifts, and movements in both (CC) and (LM) will jointly determine structural change. Movements in (CC) are governed by the following result:

Lemma 1 The factor rebalancing effect. *Assume $\sigma_2 > \sigma_1$, so that sector 2 is the flexible one. Then, an increase in k shifts the contract curve up in κ, λ-space. A proportional increase in both M_1 and M_2 shifts it down.*

Proof: The result follows immediately from the elasticities of (CC) with respect to κ, λ, k, M_1 and M_2. ■

This result is very intuitive: an increase in k or in M_1 and M_2 corresponds to a shift in the relative abundance of the two inputs, capital and effective labor. This affects relative marginal products and the optimal factor mix in each sector. For instance, for any κ, an increase in k increases the marginal product of labor more in the less flexible sector 1. Hence, as k increases, (CC) requires larger λ at any κ. Another way of thinking about this is to note that in equilibrium, a larger aggregate capital-labor ratio implies a larger wage to rental rate ratio. This leads sectors to substitute away from labor. The more flexible sector can do so more easily, implying larger λ for any given κ. The Lemma thus states that the more flexible sector tends to increase its use of the input that becomes more abundant, and reduce its use of the one that becomes relatively more scarce. We call this the *factor rebalancing effect.* This effect crucially relies on differences in substitutability across sectors.

3.2 The labor mobility condition (LM)

This condition is obtained by combining the optimal allocation of output across sectors from a consumption (final sector) and production (factor allocation) point of view. We first solve for the demand functions for the intermediate goods under perfect competition by maximizing output (1) subject to the zero profit condition $p_1Y_1 + p_2Y_2 = PY$. We normalize $P = 1$. This maximization problem yields the inverse demand functions for the intermediate inputs:

$$p_1 = \gamma \left(\frac{Y_1}{Y}\right)^{-\frac{1}{\varepsilon}}, \quad p_2 = (1-\gamma)\left(\frac{Y_2}{Y}\right)^{-\frac{1}{\varepsilon}}. \tag{9}$$

These two equations yield the relative demand, Y_1/Y_2, as a function of the relative price, p_1/p_2.

$$\frac{p_1}{p_2} = \frac{\gamma}{1-\gamma}\left(\frac{Y_1}{Y_2}\right)^{-\frac{1}{\varepsilon}} \tag{10}$$

Combining this with the condition for the optimal allocation of labor across sectors (8), which requires that $p_1 MPL_1 = p_2 MPL_2$, yields the labor mobility condition

$$LM(\kappa, \lambda, k, M_1, M_2) \equiv \frac{p_1}{p_2}\frac{MPL_1}{MPL_2} = \underbrace{\frac{\gamma}{(1-\gamma)}\left(\frac{Y_1}{Y_2}\right)^{-\frac{1}{\varepsilon}}}_{\text{Relative Price}} \underbrace{\frac{1-\alpha_1}{1-\alpha_2}\frac{(Y_1/\lambda)^{\frac{1}{\sigma_1}}}{(Y_2/(1-\lambda))^{\frac{1}{\sigma_2}}} L^{\frac{1}{\sigma_2}-\frac{1}{\sigma_1}} \frac{M_1^{\frac{\sigma_1-1}{\sigma_1}}}{M_2^{\frac{\sigma_2-1}{\sigma_2}}}}_{\text{Relative Marginal Product}} = 1. \tag{LM}$$

(Note that Y_1 and Y_2 implicitly depend on both κ and λ.) Taking κ as given, this condition describes the optimal allocation of labor across sectors in order to balance two considerations: changing the allocation of labor to a sector affects its relative marginal product, but also its relative price. As above, we will refer to the left hand side of this equation as (LM).

It is clear from (LM) that for any interior κ, allocating the entire labor endowment to one of the two sectors is not a solution. Hence, optimal λ is strictly between 0 and 1. This implies that depicted in κ, λ-space, the LM curve crosses the CC curve strictly once from above.[10]

The slope of (LM) in κ, λ-space depends on parameters. It is driven by the balance of the relative price and relative marginal product effects. Consider an increase in κ. This increases both the output and the marginal product of labor in sector 1. Increased output results in a lower relative price in this sector. For value marginal products to be equated across sectors, it is then necessary to reallocate labor to sector 2. We refer to this as the *relative price effect.* Its strength is driven by the demand elasticity ε. This is the key effect driving results in both AG and NP. At the same time, higher κ implies an increase in the marginal product of labor in sector 1, which calls for more labor to be allocated to that sector. We call this the *relative marginal product effect.* We discuss it in more detail below.

If both σ_1 and σ_2 are larger than ε, the relative price effect dominates, and allocating more capital to a sector calls for allocating more labor to the other sector. Put differently, when $\varepsilon < \sigma_1, \sigma_2$, it is easier to substitute in the intermediate goods sectors than in the final sector, and therefore the equalization of value marginal products implies a negative relationship abetwen capital and labor allocations: a downward-sloping LM curve. In the opposite case, $\varepsilon > \sigma_1, \sigma_2$, it is easier to substitute in the final goods sector. In this case, the relative marginal product effect dominates and an increase in κ requires an increase in λ, so that the LM curve is upward-sloping.

The LM curve shifts with changes in k or in productivity:

Lemma 2 Shifts in the LM curve. *For given κ, an increase in k shifts (LM) up if $\left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right) \epsilon_1 > \left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right) \epsilon_2$, where ϵ_s is the elasticity of output to capital in sector $s = \{1, 2\}$.*[11] *A proportional increase in M_1 and M_2 has the opposite effect.*

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[10] Note that there is only one intersection between the two curves since the objective function $F(Y_1(\kappa K, \lambda L), Y_2((1-\kappa)K, (1-\lambda)L))$ is strictly quasi-concave in (λ, κ) and the FOCs of maximization of a strictly quasi-concave function over a convex set yield a unique global maximum; see Takayama (1985, Theorem 1.E.3, p. 115). The strict quasiconcavity of F with respect to λ, κ results from the strict quasi-concavity of F with respect to Y_1 and Y_2 and the strict quasiconcavity of Y_1 and Y_2 with respect to their inputs.

[11] With a CES production function, ϵ_s is not a structural parameter. Each ϵ_s depends on σ_s, α_s, and also on the input allocation in the sector. In spite of this, it is still very useful to define conditions in terms of these elasticities. This is because the firm's optimality condition (7) implies that ϵ_s equals the capital income share in sector s. Given the limited variation and easy observability of factor income shares, their order of magnitude and plausible variation across sectors is in fact easier to assess than that of the structural parameters in this and many other models.

Proof: The result follows immediately from the elasticities of (LM) with respect to λ, k, M_1 and M_2. ■

The same two effects are at work here: When increasing capital in both sectors proportionally raises the relative output of sector 2 (this occurs when $\epsilon_2 > \epsilon_1$), the relative price of sector 1 increases. At the same time, the relative marginal product of labor in sector 1 increases if $\epsilon_1/\sigma_1 > \epsilon_2/\sigma_2$. This is because an increased capital input affects each sector's marginal product of labor, which is proportional to $(Y_s/L_s)^{1/\sigma_s}$, in two ways. First, for any increase in Y_s, the marginal product of labor increases more the lower σ_s, as this implies stronger complementarity between capital and labor. Hence, the relative marginal product effect favors the less flexible sector. Second, the importance of capital as an input in the sector matters. This is captured via the elasticities of sectoral output with respect to capital, ϵ_1 and ϵ_2. This reflects that given any σ, the optimal labor input increases more the more output responds to the increase in capital. Hence, the relative marginal product effect favors the sector with the lower substitution elasticity, and the sector with the higher elasticity of output with respect to capital.

Combining the two effects implies that the relative value marginal product in sector 1 increases under the condition in the Lemma, calling for more labor to be allocated to sector 1.

3.3 Development and structural change

Development, i.e. changes in k and M_s, shifts the contract curve and the labor mobility condition, leading to changes in the optimal allocation of capital and labor across sectors. Lemmas 1 and 2 give the direction of the shifts in the CC and LM curves. However, they are not sufficient to characterize changes in both λ and κ. For example, in a case where both curves shift up, it is clear that λ increases, but the response of κ is ambiguous.

We thus start by analyzing a few analytically tractable special cases in which changes in κ and λ are unambiguous. We first consider a case that illustrates the potential power of the factor rebalancing effect: here, an increase in k causes movements in κ and λ in opposite directions. This possibility, which is new to the literature, is driven by differences in substitutability across sectors. Subsequently, we briefly illustrate how, once the factor rebalancing effect is shut down, the usual prominent mechanism from the structural change literature drives factor allocations. After that, we discuss more general results and present some illustrative simulations.

3.3.1 Special case 1: A dominant factor rebalancing effect

Assume that $\sigma_2 > \sigma_1 = 1$, $\alpha_1 = \alpha_2 = \alpha$, and $\varepsilon = 1$. The (CC) and (LM) conditions simplify to

$$CC(\kappa, \lambda, k, M_2) \equiv M_2^{\frac{1-\sigma_2}{\sigma_2}} k^{\frac{\sigma_2-1}{\sigma_2}} \frac{\kappa}{(1-\kappa)^{\frac{1}{\sigma_2}}} \frac{(1-\lambda)^{\frac{1}{\sigma_2}}}{\lambda} = 1 \tag{CC'}$$

$$\begin{aligned} LM(\kappa, \lambda, k, M_2) &\equiv \frac{\gamma}{(1-\gamma)} \left(\frac{Y_2}{L}\right)^{\frac{\sigma_2-1}{\sigma_2}} \frac{(1-\lambda)^{\frac{1}{\sigma_2}}}{\lambda} M_2^{\frac{1-\sigma_2}{\sigma_2}} \qquad \text{(LM')} \\ &= \frac{\gamma}{(1-\gamma)} \left((1-\alpha)(1-\lambda)^{\frac{\sigma_2-1}{\sigma_2}} + \alpha \left(\frac{(1-\kappa)\,k}{M_2}\right)^{\frac{\sigma_2-1}{\sigma_2}} \right) \frac{(1-\lambda)^{\frac{1}{\sigma_2}}}{\lambda} = 1. \end{aligned}$$

Of course, $\sigma_2 > \sigma_1$ implies that higher k shifts up the CC curve because of the factor rebalancing effect. The parameter restrictions also pin down the slope and direction of shift of the LM curve. From Lemma 2 and the discussion preceding it, the LM curve is downward-sloping, and shifts up as k increases. Intuitively, the reduction in the price of sector 2 output calls for labor to move to sector 1. Since $\sigma_2 > \varepsilon$, the increase in the relative marginal product of labor in sector 2 is not sufficient to compensate for this.

While in the general case, the relative size of resulting shifts in CC and LM depends on parameters and allocations, the factor rebalancing effect dominates the other effects under the simplifying assumptions made here, implying that CC shifts up more than LM. [12] As a consequence, an increase in k leads to an increase in λ but a decline in κ. This result, and additional results on quantitative responses, are summarized in the following proposition:

Proposition 1 *Assume $\varepsilon = 1$, $\alpha_2 = \alpha_1 = \alpha$ and $\sigma_2 > \sigma_1 = 1$. Then the fraction of capital allocated to the less flexible sector falls as the economy's aggregate capital-labor ratio increases, while the fraction of labor in the same sector increases. Similarly, the fraction of capital (labor) allocated to the less flexible sector does not change when its level of TFP increases but it increases (decreases) with the level of TFP in the more flexible sector. More*

[12] Some intuition for the result can be obtained by consulting the conditions in Proposition 6 in the Appendix. This suggests that the key simplifying assumption $\sigma_1 = \varepsilon$ has two important implications: First, it implies that the effect of k on (LM) is always ϵ_2 times its effect on (CC), and thus smaller. Second, it implies that the required increase in λ to make each of the two conditions hold again is smaller for (LM). The reason is that with $\sigma_2 > \varepsilon$, moving labor out of sector 2 affects its relative price more than its relative marginal product.

specifically, we obtain

$$\begin{aligned}
\frac{\partial \ln \kappa}{\partial \ln k} &= -\frac{\partial \ln \kappa}{\partial \ln M_2} = \frac{(1-\sigma_2)}{\sigma_2 G(\kappa)\kappa} < 0 \qquad (11)\\
\frac{\partial \ln \lambda}{\partial \ln k} &= -\frac{\partial \ln \lambda}{\partial \ln M_2} = \left(\frac{\alpha}{1-\alpha}\right)\frac{\lambda(\kappa)}{\kappa^2}\frac{(\sigma_2-1)}{\sigma_2 G(\kappa)} > 0 \qquad (12)\\
\frac{\partial \ln \kappa}{\partial \ln M_1} &= \frac{\partial \ln \lambda}{\partial \ln M_1} = 0, \qquad (13)
\end{aligned}$$

where

$$G(\kappa) \equiv \left[\frac{1}{\sigma_2\left(1-\lambda(\kappa)\right)} + \frac{1}{\lambda(\kappa)}\right]\left(\frac{\lambda(\kappa)}{\kappa}\right)^2\left(\frac{\alpha}{1-\alpha}\right) + \left[\frac{1}{\kappa} + \frac{1}{\sigma_2(1-\kappa)}\right]$$

and $\lambda(\kappa)$ is implicitly defined by (CC'). The inequality signs in (11)-(13) are reversed when sector 1 is the more flexible one, i.e. $\sigma_2 < \sigma_1 = 1$.

Proof: See Appendix. ■

As the economy-wide capital-labor ratio increases, so does the wage to rental rate ratio. As a result, the more flexible sector substitutes away from the now more expensive input, labor, towards the relatively cheaper one, capital, at a higher rate than the less flexible sector 1 is able to do.

Changes in TFP in the Cobb-Douglas sector, M_1, leave the ratio of the wage per unit of effective labor to the rental rate, and therefore sectoral allocations, unchanged. In contrast, although increases in M_2 raise the wage to rental rate ratio, they still lower the effective cost of using labor, i.e. they reduce the wage per unit of effective labor. As a result, the more flexible sector 2, taking advantage of this change in relative factor prices, increases labor intensity of production. In both instances, cross-sectoral differences in the elasticity of substitution allow the more flexible sector to absorb a larger fraction of the relatively cheap input and to release some of the relatively expensive one.

Remark 1 *For a fixed level of productivity, an increase in the aggregate capital-labor ratio induces an increase in the wage to rental rate ratio. As a result, the capital-labor ratio increases in both sectors. This is true even when κ and λ move in different directions.*

Hence, following an increase in the aggregate capital-labor ratio, we should not expect to see declines in the *level* of the capital-labor ratio in any sector, but different, positive, growth rates.

Remark 2 *There is a level of k (let it be $\bar{k}$) such that $\kappa = \lambda$.*[13] *At this point, a* factor intensity reversal *occurs: With $\sigma_2 > \sigma_1$, $\kappa > \lambda$ for $k < \bar{k}$, and $\kappa < \lambda$ for $k > \bar{k}$.*

[13]It is clear from (CC) that this level is such that $\bar{k}^{\frac{1}{\sigma_1}-\frac{1}{\sigma_2}} = \frac{\alpha_1}{\alpha_2}\frac{1-\alpha_2}{1-\alpha_1}M_1^{\frac{1-\sigma_1}{\sigma_1}}M_2^{\frac{\sigma_2-1}{\sigma_2}}$.

An allocation with $\kappa = \lambda$ implies that the aggregate capital labor ratio and those in the two sectors are all equated. This is only possible if the contract curve coincides with the 45-degree line. (CC) defines the level of k for which this occurs. Given $\sigma_2 > \sigma_1$, the factor rebalancing effect implies that an increase in k above this level shifts the contract curve up in κ, λ-space. As a result, it must be that $\lambda > \kappa$ at the optimal allocation if $k > \bar{k}$. The reverse argument applies for $k < \bar{k}$.[14]

Intuitively, above $\bar{k}$, capital is relatively abundant, and its rental price relatively low. The more flexible sector is freer to respond to this, implying that it uses it more intensively, so that $\kappa < \lambda$ in this range. Below $\bar{k}$, capital is relatively scarce, so its rental price is relatively high. The more flexible sector can more easily substitute to labor, implying $\kappa > \lambda$ in this range. One could thus say that at $\bar{k}$, the economy's endowments of capital and labor are "balanced" given the sectoral production technologies.

Finally, we can characterize the behavior of the relative price, relative output, and relative value added of the two sectors.

Lemma 3 Relative price, relative output, relative value added. *The relative price p_1/p_2 declines in k for $k < \bar{k}$, is minimized at $\bar{k}$, and increases for $k > \bar{k}$. Relative output Y_1/Y_2 increases in k for $k < \bar{k}$, is maximized at $\bar{k}$, and decreases for $k > \bar{k}$. Since $\varepsilon = 1$ by assumption, relative value added $p_1Y_1/(p_2Y_2)$ remains constant at $\gamma/(1-\gamma)$.*

Proof. See Appendix. ■

At $k = \bar{k}$, the relative price of sector 1 output is minimized, because its relative marginal cost is minimized at this point. From the discussion of Remark 2, when k is larger or smaller than $\bar{k}$, k is relatively abundant or scarce. This imbalance affects the marginal cost of the less flexible sector more, implying that p_1/p_2 increases as k moves away from $\bar{k}$. In the special case under consideration, since the final sector is Cobb-Douglas, the change in the relative price is accompanied by a counteracting change in relative output that leaves relative value added unchanged. If instead the products of the two intermediate sectors are gross complements (substitutes), i.e. if $\varepsilon < (>)1$, the value added share of the less flexible sector increases (decreases) as k moves away from $\bar{k}$. Hence, a general version of the model predicts that changes in k generate structural change in terms of both inputs and value added.

Before showing results for a more general setting, we briefly show next that when the factor rebalancing effect is shut down, as commonly is the case in previous work on structural change, we recover standard results from that literature.

[14]Remark 2 holds not only under the assumptions of Proposition 1, but in the general model.

3.3.2 Special case 2: Differences in factor intensity (AG)

Let $\sigma_1 = \sigma_2 = 1$ and $\alpha_2 > \alpha_1$ so that sector 1 is labor-intensive, as in AG. As shown in Section 3.1, the assumption of sectoral Cobb-Douglas technologies shuts down the factor rebalancing effect. Therefore, changes in the aggregate capital-labor ratio or in sectoral productivities shift only the LM curve, tracing out points along the unchanged CC curve. As a result, the fractions of capital and labor allocated to any of the two sectors always move in the same direction. When intermediate goods are complements, $\varepsilon < 1 = \sigma_s$, the relative price effect dominates the marginal product effect as the aggregate capital-labor ratio increases. As $\alpha_2 > \alpha_1$ directly implies $\epsilon_2 > \epsilon_1$ here, this implies that the LM curve shifts up and the labor intensive sector increases its shares of capital and labor. The following proposition summarizes the quantitative responses of sectoral factor allocations to changes in the aggregate capital-labor ratio and sectoral productivities:

Proposition 2 *Assume $\sigma_1 = \sigma_2 = 1, \alpha_2 > \alpha_1$ and $\varepsilon < 1$. The fractions of capital and labor allocated to the labor-intensive sector increase as the aggregate capital-labor ratio increases:*

$$\begin{aligned}
\frac{d\ln\kappa}{d\ln k} &= \frac{(1-\varepsilon)(\alpha_2-\alpha_1)(1-\kappa)}{1+(1-\varepsilon)(\alpha_2-\alpha_1)(\kappa-\lambda)} > 0 \\
\frac{d\ln\lambda}{d\ln k} &= \frac{d\ln\kappa}{d\ln k}\frac{\alpha_1}{\alpha_2}\frac{(1-\alpha_2)}{(1-\alpha_1)}\frac{\lambda}{\kappa} > 0. \\
\frac{d\ln\kappa}{d\ln M_2} &= -\frac{d\ln\kappa}{d\ln M_1}\frac{(1-\alpha_2)}{(1-\alpha_1)} = \frac{(1-\varepsilon)(1-\kappa)(1-\alpha_2)}{1+(1-\varepsilon)(\alpha_2-\alpha_1)(\kappa-\lambda)} > 0 \\
\frac{d\ln\lambda}{d\ln M_2} &= -\frac{d\ln\lambda}{d\ln M_1}\frac{(1-\alpha_2)}{(1-\alpha_1)} = \frac{d\ln\kappa}{d\ln M_2}\frac{\alpha_1}{\alpha_2}\frac{(1-\alpha_2)}{(1-\alpha_1)}\frac{\lambda}{\kappa} > 0
\end{aligned}$$

The reverse is true when $\varepsilon > 1$.

The intuition for this proposition is best understood by considering the effects of capital deepening in the absence of sectoral reallocations. With unchanged λ and κ, higher k implies that output of the capital-intensive sector 2 grows more than that of sector 1.[15] When sectoral outputs are gross complements (substitutes) in the production of the final good, i.e. when $\varepsilon < 1$ $(\varepsilon > 1)$, the relative price then shifts against (in favor of) the fast-growing capital-intensive sector. This change in relative prices implies that equating marginal revenue products of factors across sectors requires allocating a larger (smaller) fraction of resources to the labor-intensive sector 1.

[15] This is reminiscent of the Rybczinski Theorem in international trade theory.

3.3.3 Special case 3: Differences in relative productivity (NP)

Let $\sigma_1 = \sigma_2 = 1$ and $\alpha_1 = \alpha_2 = \alpha$, as in NP. In this case the factor rebalancing effect is shut down, as in AG. The CC curve always coincides with the 45-degree line, and sectoral and aggregate capital-labor ratios are identical. In addition, equal sectoral factor intensities imply that the relative price and relative marginal product effects are absent for changes in k and for proportional changes in M_1 and M_2. Only changes in relative productivity shift the LM curve. For instance, when $\varepsilon < 1$, increases in M_2/M_1 are associated with a positive relative price effect that shifts the LM curve up. As a result the fractions of capital and labor allocated to the (technologically) laggard sector 1 increase. The following proposition summarizes these results.

Proposition 3 *Assume $\sigma_1 = \sigma_2 = 1$, $\alpha_2 = \alpha_1 = \alpha$, and $\varepsilon < 1$. Then the fractions of capital and labor allocated to the laggard sector, sector 1, increase as its relative level of total factor productivity decreases, i.e. as sector 1's TFP falls relative to sector 2's TFP.*

$$\begin{aligned}
\frac{d\ln\kappa}{d\ln k} &= \frac{d\ln\lambda}{d\ln k} = 0 \\
\frac{d\ln\lambda}{d\ln M_2} &= \frac{d\ln\kappa}{d\ln M_2} = -\frac{d\ln\lambda}{d\ln M_1}\frac{M_2}{M_1} = -\frac{d\ln\kappa}{d\ln M_1}\frac{M_2}{M_1} \\
&= \left(\frac{1-\gamma}{\gamma}\right)^{\varepsilon}(1-\alpha)(1-\varepsilon)\left(\frac{M_2}{M_1}\right)^{(1-\alpha)(\varepsilon-1)}\lambda>0
\end{aligned}$$

The reverse is true when $\varepsilon > 1$.

The intuition for these results is analogous to that for Proposition 2.

3.3.4 The general case: Factor rebalancing versus the relative price effect

Similar results to Propositions 1, 2 and 3 can be derived for more general parameter configurations. These results are collected in Propositions 6 and 7 in the Appendix. In this section, we discuss results for some particular parameter configurations with the objective to illustrate possible outcomes and their drivers. We first do so using the CC and LM curves, and then show paths of key outcomes as a capital accumulates.

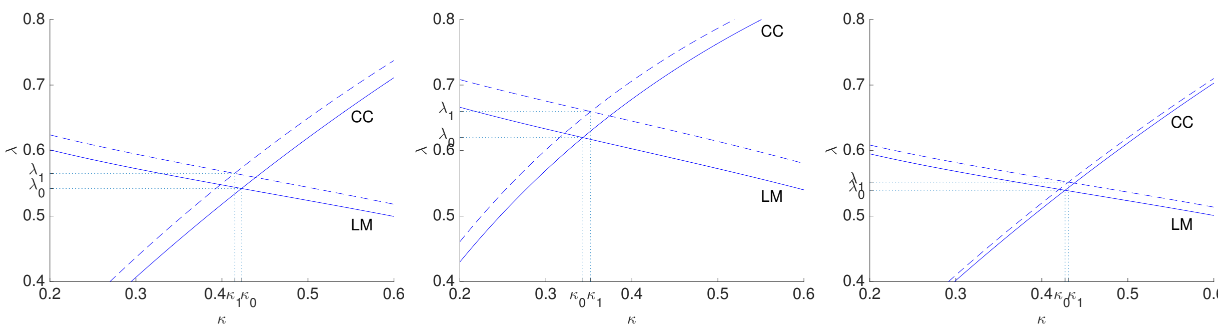
Figure 2 illustrates the effect of an increase in the aggregate capital-labor ratio in three situations.[16] Each situation features all three effects discussed above, but with outcomes that differ due to varying strength of the effects. First, we assume as before that $\sigma_2 > \sigma_1$ in all panels, so that the CC curve shifts up because of the factor rebalancing effect. Second,

[16]All panels show the equilibrium reaction to raising the aggregate capital labor ratio from 1 to 2. Parameter values are given in the note to the figure.

the stronger complementarity between capital and labor in sector 1 also implies that the relative marginal product effect moves the LM curve up. Third, recall that in general, the relative price effect moves the LM curve up if $\epsilon_1 < (>) \epsilon_2$ and $\varepsilon < (>) 1$, and has no effect if $\varepsilon = 1$ (as in Proposition 1) or if $\epsilon_1 = \epsilon_2$. Contrary to Proposition 1, $\varepsilon < 1$ in all panels of Figure 2, giving more strength to the relative price effect.

Still, the overall parameter configuration is chosen such that in the left panel, ϵ_2 only slightly exceeds ϵ_1, implying that a fairly weak relative price effect shifts LM up by only a little. As a result, the shift in CC is dominant, leading to the same outcome as in the case in Proposition 1: λ increases but κ declines. In the middle panel, α_2 is raised to a point where ϵ_2 exceeds ϵ_1 substantially. As a consequence, the relative price effect becomes so strong that the shift in LM exceeds that in CC, and λ and κ both increase. Finally, the right panel illustrates how the same outcome can occur if the difference between σ_1 and σ_2 is smaller. By weakening the factor rebalancing effect more than the relative marginal product effect, this change leads to a smaller shift in CC relative to LM.

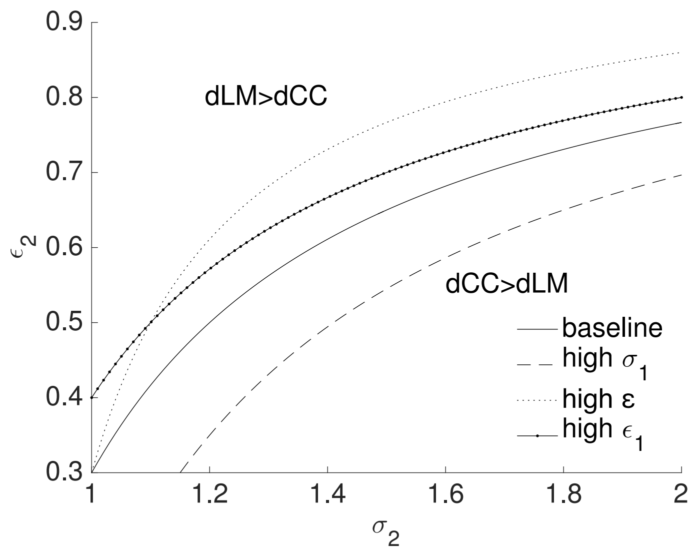
Figure 2: The effect of higher k on the allocations of capital and labor – three examples



Parameter values: Left (baseline): $\gamma = 0.5$, $\varepsilon = 0.5$, $\sigma_1 = 1$, $\sigma_2 = 1.2$, $\alpha_1 = 0.3$, $\alpha_2 = 0.4$, $M_1 = M_2 = 1$. Middle: $a_2 = 0.55$. Right: $\sigma_2 = 1.05$. Solid lines: $k = 1$. Dashed lines: $k = 2$.

Figure 3 shows the same relationships in a different way. Each line shows, for a given combination of ε, σ_1 and ϵ_1, for which combinations of σ_2 and ϵ_2 the LM and CC curves shift by the same amount in reaction to higher k. Hence, for combinations of σ_2 and ϵ_2 on the line, λ increases but κ is unchanged. The LM curve shifts up more (less) than the CC curve for ϵ_2 above (below) the line or for σ_2 left (right) of the line, implying that κ increases (decreases). (λ increases in all cases shown.) The different lines show how these relationships vary with parameters. First, higher σ_1 implies a smaller difference in substitutability for any level of σ_2, and thus a weaker factor rebalancing effect, as also seen in the right panel of Figure 2. As a result, the shift in LM dominates that in CC already for smaller values of ϵ_2. Second, higher ε makes the relative price effect less powerful, implying that larger ϵ_2 is required for it to push LM more than CC. Finally, higher ϵ_1 implies that larger ϵ_2 is required for the relative price effect to attain a given strength.

Figure 3: Size of shift in LM versus CC curves



Note: dLM (dCC) denotes size of upward shift in the LM CC curve. Parameter values used: baseline: $\varepsilon = 0.5$, $\sigma_1 = 1$, $\epsilon_1 = 0.3$. Higher $\sigma_1 = 1.15$. Higher $\varepsilon = 0.75$. Higher $\epsilon_1 = 0.4$.

Finally, note that when increases in k do not cause a relative price effect because $\epsilon_2 = \epsilon_1$, the factor rebalancing effect generally dominates the relative marginal product effect. This implies that the shift in LM can dominate that in CC only when the relative price effect is present. One can thus think of equilibrium outcomes in the model as being determined by a competition between the factor rebalancing effect and the relative price effect. Previous work on structural change has focussed on the relative price effect only. Notably, in AG and NP, where the assumption $\sigma_1 = \sigma_2$ closes down the factor rebalancing effect, only the relative price effect matters, and only LM shifts. Here, in contrast, we can address cases where there is no relative price effect, and also cases in which all effects are present.

3.3.5 The general case: Illustrative simulations

Next, we present some simulations that show how key variables evolve with k. We do so for the same parameter values as in the first two panels of Figure 2 above. These values are in the broad range of reasonable values. Finally, we also briefly report results for a value of ε of 1.1. In this case, the simulations complement the analytical results because here, ε lies between σ_1 and σ_2, a case that is not covered by our analytical results in the Appendix.

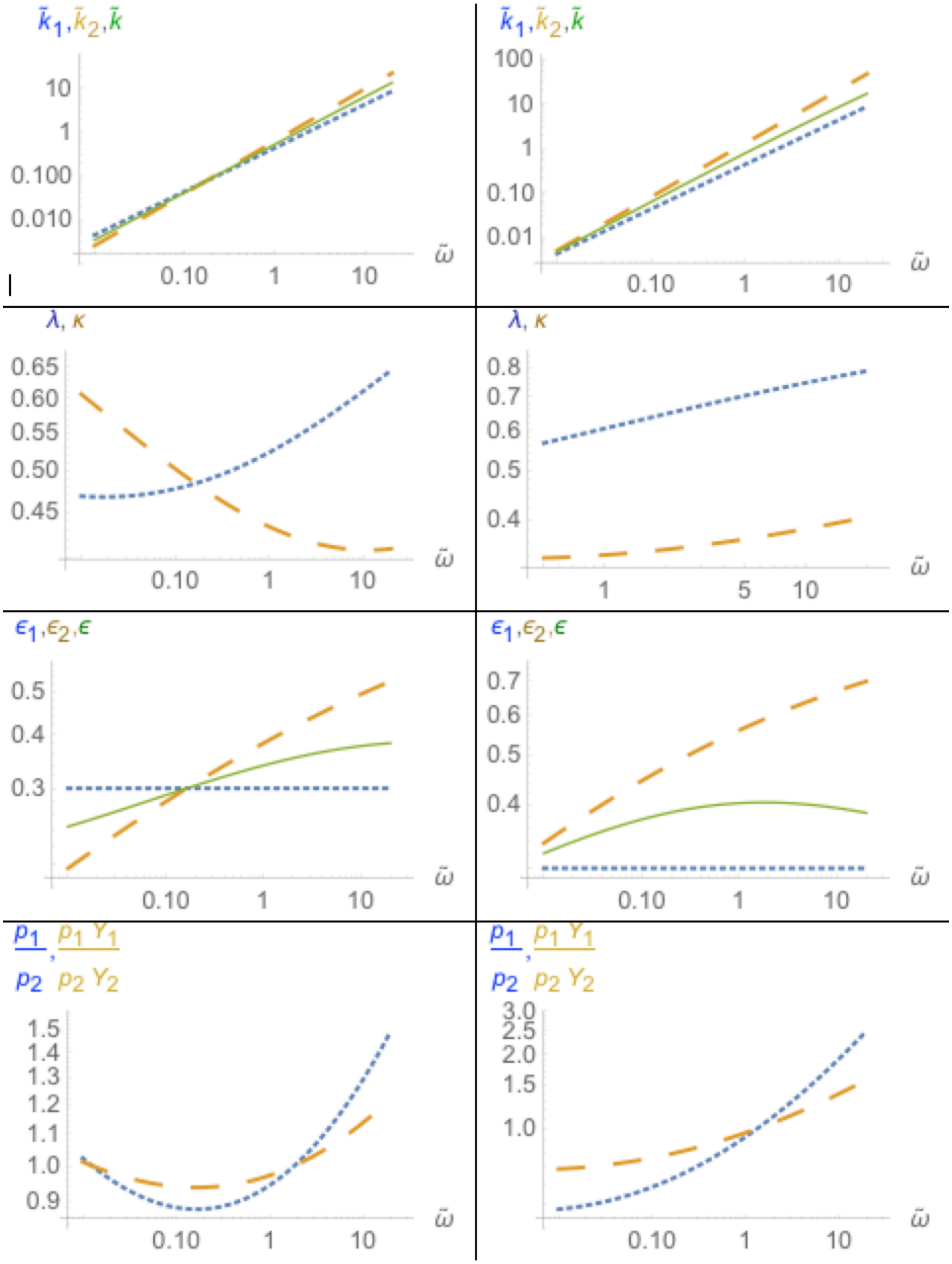
It turns out to be useful to track the evolution of key variables in terms of the relative input price. For common M_s, this is given by $\tilde{\omega} \equiv w/(MR)$. From equations (7) and (8), sectoral ratios of capital to effective labor, $\tilde{k}_s$, depend on $\tilde{\omega}$ via $\tilde{k}_s(\tilde{\omega}) = (\tilde{\omega}\alpha_s/(1-\alpha_s))^{\sigma_s}$. It is clear from this expression that sectoral capital-labor ratios grow at different rates as the relative input price changes, with the difference being determined by the ratio of sectoral substitution elasticities.

Figure 4 presents the results of these simulations. Since there is a monotonically increasing relationship between the relative input price and aggregate capital per unit of effective labor, as shown in the top two panels, one can think of the x-axis in the last six panels of this figure in terms of the latter variable.[17] The left panels illustrate the baseline case in Figure 2 where the factor rebalancing effect is dominant. As the relative input price increases, the flexible sector is in a better position to take advantage of this decrease in the relative price of capital and, therefore, the fraction of capital (labor) that it employs increases (falls). When effective labor is relatively abundant, for instance $\tilde{\omega} = 0.1$, the flexible sector employs roughly 53% of the labor force and only 50% of the capital stock. This situation is reversed as effective labor becomes relatively scarce, and thus more expensive. For instance, as $\tilde{\omega}$ increases to 1, the flexible-sector fraction of labor falls by more than 5 percentage points, while its fraction of the capital stock increases by almost 7 percentage points. As a result, the flexible sector is more labor-intensive when the aggregate ratio of capital to units of effective labor is low, i.e. effective labor is abundant, and more capital-intensive when the aggregate ratio of capital to effective labor is high. Given our parameter choices, this capital-intensity reversal takes place when aggregate capital per unit of effective labor is 1/12.

The third panel of Figure 4 reports the evolution of the capital income shares, or elasticities of (sectoral) output with respect to capital. The relevant benchmark for understanding this evolution is given by the Cobb-Douglas case. With unitary sectoral elasticities of substitution, $\sigma_s = 1$, factor income shares are independent of relative factor prices. This is the case for sector 1 in our calibration. If the elasticity of the flexible sector exceeds one, as is the case for sector 2 here, $\tilde{k}_s(\tilde{\omega}) = (\tilde{\omega}\alpha_s/(1-\alpha_s))^{\sigma_s}$ implies that an increase in the relative input price leads to a more than proportional increase in capital per unit of effective labor. As a result the capital income share in the flexible sector, ϵ_2, increases as capital accumu-

[17] An additional issue concerns the range of $\tilde{\omega}$ we consider in our numerical exercise. We report values in the range $\tilde{\omega} \in [0.1, 20]$. To get a sense of the meaning of this range, consider a Solow model with $s = 0.2, \delta = 0.05$ and $m = n = 0$. Here, if we move from 10% to 200% of the steady state capital stock, the relative input price increases from 2 to 17.5. If the same exercise is conducted assuming $m = 0.02$ and $n = 0.01$, the relative input price for the same range of the steady state capital per unit of effective labor goes from 2.5 to 16.

Figure 4: Model simulations



Notes: Parameter settings for the simulations as in Figure 2, first two panels. First variable (dotted line), second variable (dashed line), third variable (solid line). Relative input price $\tilde{\omega} = \omega/(RM)$ on the horizontal axis of each graph. Both axes are on a log scale in each graph.

lates. The aggregate capital income share is an average of the sectoral capital income shares, weighted by sectoral value added shares.

The last panel in the left column of Figure 4 presents the evolution of the relative price of the two sectors' output and of the share of value added produced in sector one. Recall that the two are related by the condition for optimal demand from the final goods sector. The path of the relative sectoral price is non-monotonic, with a minimum at the capital-intensity reversal, as indicated by Lemma 3. At this point, the cost advantage enjoyed by the more flexible sector due to its higher elasticity of substitution vanishes, and sectoral levels of capital per unit of effective labor coincide. As capital per unit of effective labor increases above (falls below) this point, the flexible sector takes advantage of the change in the relative input price, reducing its unit costs relative to those of sector 1. As a result, the relative sectoral price increases as aggregate capital per unit of effective labor moves away from $1/12$, in either direction. Since sectoral outputs are gross complements in the production of final output, changes in the relative sectoral price induce less than proportional changes in relative quantities. As a result, the path of relative value added – a common measure of structural change – is similar to that of the relative sectoral price.

The right panels of Figure 4 illustrate the case where the relative price effect dominates, corresponding to the middle panel in Figure 2. The evolutions of the ratios of capital to effective labor, capital income shares, relative price, relative value added are qualitatively similar to those in the left column. Nonetheless, in this case, the dominant relative price effect induces the less flexible sector to absorb increasing fractions of both inputs as capital accumulates and the terms of trade turn in its favor. The different sectoral elasticities of substitution express themselves through a different rate of absorption of the two inputs, with the fraction of labor in the less flexible sector growing faster than the fraction of capital.

Finally, we explored a case where the sectoral substitution elasticities lie on both sides of the final output one, $\sigma_1 < \varepsilon < \sigma_2$. The qualitative patterns coincide with those described above. Nonetheless, since sectoral outputs are gross substitutes in the production of final output in this example, the relative sectoral price moves in the opposite direction of relative value added. The results of these simulations are reproduced in Figure 8 in the Appendix.

Overall, the patterns emerging from these numerical exercises are exactly in line with our analytical results. They also show that in addition to factor reallocations, there is structural change in terms of value added.

3.4 The aggregate elasticity of substitution

Evaluating the response of the economy-wide capital-labor ratio to changes in the factor-price ratio requires a measure of the degree of substitutability between capital and labor at the aggregate level. In recent years, several studies have estimated this aggregate elasticity using U.S. time series data (see e.g. Antras (2004), Herrendorf et al. (2015), Leon-Ledesma et al. (2015)), cross-sectional U.S. data (Oberfield and Raval, 2014) or cross-country data (Karabarbounis and Neiman, 2014). In our two-sector framework, this elasticity is not a "deep" parameter, but depends on other parameters and on allocations. In what follows, we borrow from the dual approach developed in Jones (1965) to characterize this aggregate elasticity.

Lemma 4 The aggregate elasticity of substitution. *The aggregate elasticity of substitution, σ, is a weighted average of the three primary elasticities: the elasticity of substitution between the two intermediate inputs in the production of the final good, ε, and the two sectoral elasticities, σ_1 and σ_2:*

$$\sigma = \gamma_0 \varepsilon + \gamma_1 \sigma_1 + \gamma_2 \sigma_2 \tag{14}$$

where the weights,

$$\begin{aligned}
\gamma_0 &\equiv (\epsilon_2 - \epsilon_1)(\lambda - \kappa) \\
\gamma_1 &\equiv \lambda \epsilon_1 + \kappa (1 - \epsilon_1) \\
\gamma_2 &\equiv (1 - \lambda) \epsilon_2 + (1 - \kappa)(1 - \epsilon_2)
\end{aligned}$$

add up to one and, as before, ϵ_s is the capital income share in sector s.

Proof: See Appendix. ■

Although the primary elasticities are constant, in general the aggregate elasticity of substitution varies with the sectoral composition of output. As an exception, it is constant if $\sigma_1 = \sigma_2 = 1$ and $\epsilon_1 = \epsilon_2 = \alpha$. Intuitively, if factor income shares in both sectors are equal, then $\gamma_0 = 0$, so that the aggregate elasticity of substitution is independent of the elasticity of substitution of the final sector. As a result, the aggregate elasticity reduces to a weighted average of the sectoral elasticities. It is constant when both sectoral elasticities coincide. When $\sigma_1 = \sigma_2 = 1$ but the capital-intensity of the two sectors differs (as e.g. in AG), the aggregate elasticity is given by

$$\sigma = 1 + (\alpha_2 - \alpha_1)(\lambda - \kappa)(\varepsilon - 1) < 1 \Longleftrightarrow \varepsilon < 1. \tag{15}$$

In this case the aggregate elasticity, which changes with the process of structural change, is below one if the elasticity of substitution between inputs in the final sector is below one. This arises since (14) reduces to a weighted average of the elasticity in the final sector, ε, and 1.

Finally, under the assumptions in Proposition 1, i.e. $\sigma_1 = \varepsilon = 1$ and $\sigma_2 \neq 1$, the aggregate elasticity of substitution is given by

$$\sigma = \gamma_0 + \gamma_1 + \gamma_2\sigma_2 > 1 \Longleftrightarrow \sigma_2 > 1. \tag{16}$$

Here, σ is a weighted average of 1 and one of the sectoral elasticities, σ_2, so that σ_2 determines whether the aggregate elasticity lies above or below one. Since γ_2 varies with development, here, too, the aggregate elasticity of substitution is not constant.

4 The dynamic problem

The previous section has analyzed structural change for arbitrary, one-time changes in k or M. In this section, we turn to the characterization of the solution for the full dynamic problem. For tractability, we are forced to focus on the particular case for which we have shown results in Proposition 1.[18] Recall that this case arises when $\varepsilon = \sigma_1 = 1$, and σ_2 is strictly larger or smaller than one. In addition, sectors are symmetric in all other respects, i.e. $\alpha_2 = \alpha_1 = \alpha$ and $M_1(t) = M_2(t) = M(t)$, which implies $m_1 = m_2 = m$. Results thus are driven purely by the difference in substitutability.

We define aggregate capital per unit of effective labor,

$$\chi\left(t\right) \equiv \frac{K\left(t\right)}{L\left(t\right)M\left(t\right)} \tag{17}$$

Then, using (3) we reach,

$$\widehat{\chi} \equiv \frac{\dot{\chi}\left(t\right)}{\chi\left(t\right)} = \frac{\dot{K}(t)}{K(t)} - n - m = \frac{I(t)}{K(t)} - n - m - \delta = s\frac{Y(t)}{K(t)} - n - m - \delta \tag{18}$$

Combining the CC curve, (7), (9) and (17), we reach the following one-to-one relationship between χ and κ,

$$\chi\left(\kappa\right) = \left(\gamma(1-\alpha)\right)^{\frac{\sigma_2}{\sigma_2-1}} \frac{\left(1-\kappa\right)^{\frac{1}{\sigma_2-1}}}{\left(\kappa-\alpha\gamma\right)\left(\kappa\left(1-\gamma(1-\alpha)\right)-\alpha\gamma\right)^{\frac{1}{\sigma_2-1}}} \tag{19}$$

[18] See the working paper version of this paper for results on the dynamics of the AG and NP models.

where $\chi'(\kappa) < 0$ (resp. $\chi'(\kappa) > 0$) for all $\kappa \in (\underline{\kappa}, 1)$ if $\sigma_2 > 1$ (resp. $\sigma_2 < 1$), where $\underline{\kappa} = \alpha\gamma/[1-\gamma+\alpha\gamma]$. Furthermore, it is worth noticing that when $\sigma_2 > 1$, $\chi(1) = 0$ and $\lim_{\kappa\to\underline{\kappa}} \chi(\kappa) = \infty$, and when $\sigma_2 < 1$, $\chi(\underline{\kappa}) = 0$ and $\lim_{\kappa\to 1} \chi(\kappa) = \infty$.

The following proposition summarizes the dynamic behavior of this model.

Proposition 4 *Under the stated assumptions, given the initial condition $\chi(0) = \chi_0$, the competitive equilibrium path satisfies the following differential equation:*

$$\dot{\kappa} = \frac{vD\pi(\kappa) - (\delta+m+n)}{H(\kappa)} \tag{20}$$

where

$$D \equiv \left[\frac{1-\gamma}{\gamma}\right]^{\frac{(1-\gamma)\sigma_2}{\sigma_2-1}} (\gamma(1-\alpha))^{-\frac{(1-\alpha)\gamma}{\sigma_2-1}}, \tag{21}$$

$$\pi(\kappa) \equiv \frac{\kappa^{(\sigma_2-\gamma)/(\sigma_2-1)} (\kappa(1-\gamma(1-\alpha)) - \alpha\gamma)^{\frac{\gamma(1-\alpha)}{\sigma_2-1}}}{(1-\kappa)^{\frac{1-\alpha\gamma}{\sigma_2-1}}}, \tag{22}$$

and

$$H(\kappa) \equiv -\left(\frac{1}{\sigma_2-1}\right)\left[\frac{(1-\gamma+\alpha\gamma)(\kappa-\alpha\gamma) + \sigma_2(\kappa(1-\gamma(1-\alpha)) - \alpha\gamma)}{(\kappa-\alpha\gamma)(\kappa(1-\gamma(1-\alpha)) - \alpha\gamma)}\right] \tag{23}$$

where $H(\kappa) < 0$ (resp. $H(\kappa) > 0$) for all $\kappa \in (\underline{\kappa}, 1)$ if $\sigma_2 > 1$ (resp. $\sigma_2 < 1$).

Proof. See Appendix. ■

Let's turn now to the characterization of the constant growth path.

Proposition 5 *Under the stated assumptions, there exists a unique (non-trivial) CGP that satisfies*

$$\begin{aligned}
\pi(\kappa^{ss}) &= \frac{\delta+m+n}{vD}, \qquad \lambda^{ss} = \frac{\gamma(1-\alpha)\kappa^{ss}}{\kappa^{ss}-\alpha\gamma}, \\
\chi^{ss} &= (\gamma(1-\alpha))^{\frac{\sigma_2}{\sigma_2-1}} \frac{(1-\kappa^{ss})^{\frac{1}{\sigma_2-1}}}{(\kappa^{ss}-\alpha\gamma)(\kappa^{ss}(1-\gamma(1-\alpha)) - \alpha\gamma)^{\frac{1}{\sigma_2-1}}} \\
g^{ss} &= z^{ss} = g_1^{ss} = g_2^{ss} = z_1^{ss} = z_2^{ss} = n+m \qquad n = n_1^{ss} = n_2^{ss}.
\end{aligned}$$

The steady state associated with this CGP is locally stable.

Proof. Notice that (20) is an autonomous differential equation with a unique (non-trivial) steady state, κ^{ss}. The CGP associated with this steady state is locally stable since, evaluated at that point, $\frac{\partial\dot{\kappa}}{\partial\kappa} = \frac{vD\pi'(\kappa^{ss})}{H(\kappa^{ss})} < 0$. ■

This result has several interesting implications. First, since the sectoral TFP growth rates are identical, both sectors grow at the same rate along the CGP which, of course, is the same as the growth rate of the aggregate economy. Second, the capital-output ratio and the rental rate are constant along the CGP, and so is the share of capital in national income, while the wage rate grows at the exogenous rate of TFP growth, m. Finally, once the economy reaches the CGP the process of sectoral reallocation comes to an end. This is the case since along such a path capital per unit of effective labor, χ, is constant and therefore the incentives for sectoral reallocations induced by capital deepening and technological change perfectly cancel out, since they are exactly equal but work in opposite directions. This becomes clear once one notices that (11)-(13) imply that $\frac{\partial \lambda}{\partial k}\frac{k}{\lambda} = -\frac{\partial \lambda}{\partial M}\frac{M}{\lambda}$ and $\frac{\partial \kappa}{\partial k}\frac{k}{\kappa} = -\frac{\partial \kappa}{\partial M}\frac{M}{\kappa}$.[19]

During the transition to a balanced growth path, differences in capital-labor substitutability across sectors can thus lead to sectoral reallocations. For instance, during a transition "from below", along which capital, K, grows faster than effective labor, ML, the more flexible sector will substitute towards capital, the input that becomes relatively abundant. Hence, the more flexible sector will become more capital intensive, and the less flexible sector more labor intensive. As a consequence, the capital-labor ratio in the more flexible sector will grow faster than its counterpart in the less flexible sector.

It is straightforward to extend these results to a Ramsey-Cass-Koopmans framework of optimal saving. In this case, an Euler equation replaces the exogenous saving rate, and the intertemporal elasticity of substitution plays no role since only final output is used for both consumption and capital accumulation. We refer the interested reader to the working paper version of this paper for additional details.

5 Quantitative application: structural change out of agriculture

In the introduction, we mentioned that in the U.S., structural change out of agriculture was accompanied by significantly faster growth of the capital-labor ratio in agriculture compared to the rest of the economy. At the same time, the share of value added produced in agriculture declined from 10% to around 1% of GDP.

Clearly, structural change has not only taken place in the U.S.. In this section, we first show some suggestive evidence on the relationship between sectoral allocations and capital

[19]The same conclusion could be reached in terms of the ratio of the wage per unit of effective labor to the rental rate that drives sectoral reallocations. Once χ is constant, this ratio is also constant, so structural change stops.

Table 1: Sectoral capital intensity across countries

	Y/L	K^{na}/L^{na}	K^a/L^a	L^a/L	K^a/K
Rich 5	18,000	82,273	100,318	5.0%	5.7%
Poor 5	807	6,482	191	78.2%	10.8%
Ratio	22	13	525	1/17	1/2
Mean	8,239	36,954	26,790	28.3%	7.4%
Std. Dev.	6,068	29,346	33,992	23.6%	3.7%
Coeff. Var.	0.74	0.19	1.27	0.84	0.5
Min	529	1,297	23	2.0%	1.8 %
Max	20,000	99,492	125,621	84.0%	16.4%

Source: Crego et al. (2000), Duarte and Restuccia (2010) and GGDC Total economy database.

intensity across countries. We then proceed to a quantitative analysis of the U.S. experience, illustrating the importance of differences in capital-labor substitutability relative to other drivers of structural change.

5.1 Suggestive cross-country evidence on capital intensity and structural change out of agriculture

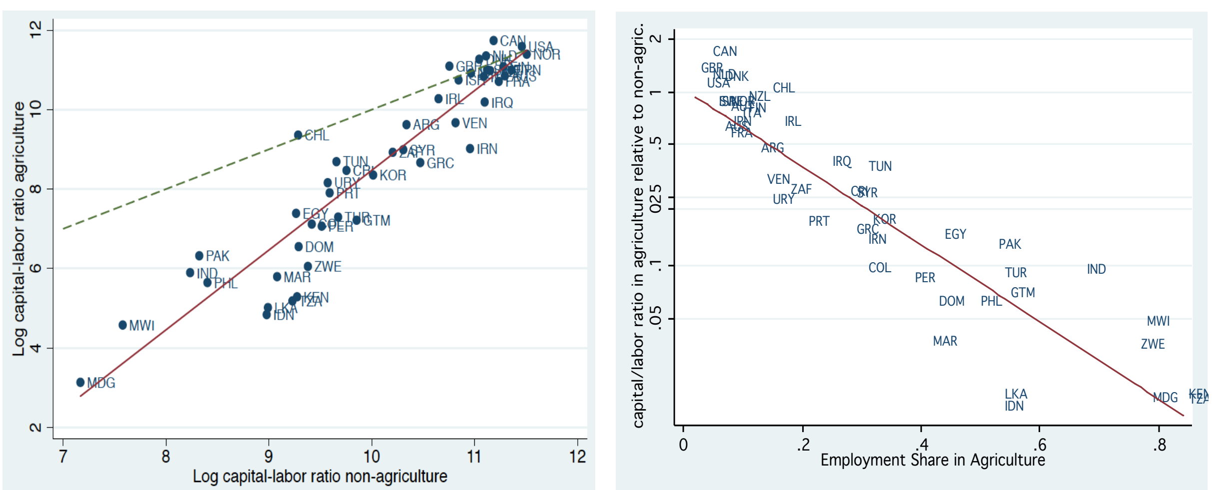
The process of economic development is always and everywhere characterized by substantial reallocations of resources out of agriculture. As a result of this process, the differences in sectoral structure between developed and developing countries are staggering. On the one hand, rich countries, such as the U.S., the U.K. or Belgium employ less than 3% of their labor force in agriculture, while on the other hand, poor countries such as Nepal, Burundi or Niger have employment shares in agriculture in excess of 90%.

These differences in employment shares are compounded by large differences in labor productivity and capital intensity. As Restuccia, Yang, and Zhu (2008), Chanda and Dalgaard (2008), and Gollin, Lagakos, and Waugh (2014) report, the differences in agricultural labor productivity between rich and poor countries are twice as large as those in aggregate labor productivity. Mundlak (2000) finds that the cross-country distributions of various measures of investment and capital show much larger dispersion in agriculture compared to the rest of the economy.

This is clear in Table 1, which combines data on sectoral capital stocks for 50 countries collected by Crego et al. (2000) with data on labor allocations from Duarte and Restuccia

(2010) to illustrate the cross-country variation in sectoral capital-labor ratios. The table shows that in rich countries, agriculture is on average somewhat more capital-intensive than the rest of the economy, while in the poorest economies, agriculture uses very little capital at all.[20] The size of these differences in capital per worker across sectors and countries is stunning. In the 5 richest countries in our sample, capital per worker outside of agriculture is 13 times larger than in the 5 poorest countries. At the same time, it is 500 times larger in the agricultural sector.[21] Figure 5(a) plots capital-labor ratios in agriculture versus those in non-agriculture. Since the correlation coefficient between non-agricultural capital-labor ratios and income per capita is above 0.95, one can think of this variable as a measure of development. A simple regression of agricultural capital per worker on its non-agricultural counterpart suggests that a one percentage point increase in the non-agricultural capital-labor ratio is associated with a two percentage point increase in agricultural capital per worker. The relative capital intensity of agriculture thus increases with development.

Figure 5: Sectoral capital labor ratios, structural change, and development



(a) Sectoral capital-labor ratios: agriculture vs. non-agriculture

(b) Relative capital-labor ratios vs. agricultural employment share

Notes: Data sources: Crego et al. (2000) and Duarte and Restuccia (2010). In both panels, the solid line is an OLS regression line. The dashed line in panel (a) is the 45-degree line.

Figure 5(b) relates the same pattern more closely to structural change. It plots the

[20] The five richest countries in our sample are the U.S., Canada, Denmark, Norway and Sweden, while the five poorest ones are Tanzania, Malawi, Madagascar, Kenya and India. Data is for 1990.

[21] An example may help visualizing how these differences come about. For instance, the Lexion 590R, the world's largest combine harvester, has the capacity to harvest 1,800 bushels of wheat per hour. This capacity is equivalent to 540 man-hours.

capital-labor ratio in agriculture relative to that in non-agriculture (on a log scale) against the fraction of persons engaged in agriculture. The relationship is very clearly negative. A regression of the log relative capital-labor ratio on the share of employment in agriculture shows that a decline in the share of employment by one percentage point goes along with an increase in the relative capital-labor ratio by 5 percent. This implies that the relative capital-labor ratio doubles every time the employment share in agriculture declines by 14 percentage points. As agriculture loses importance as a share of employment, it becomes more capital intensive relative to the rest of the economy.

Given the important cross-country variation in wage to rental rate ratios, one may interpret the large sectoral variation in capital-labor ratios as the result of a relatively high elasticity of substitution between inputs in agriculture. This interpretation is consistent with the evidence provided by Herrendorf et al. (2015). These authors estimate sectoral CES production functions using postwar U.S. data. They report an estimate of the elasticity of substitution between capital and labor in agriculture of 1.58, twice as high as their estimates for manufacturing and services. Rosenzweig (1988) implicitly acknowledges this substitution capability of agricultural production when arguing that obstacles to migration out of agriculture depress rural wages, inducing farmers to substitute cheap labor for capital and intermediate inputs. Along similar lines, Manuelli and Seshadri (2014) provide evidence on the impact of low labor costs on the slow rate of adoption of tractors in U.S. agriculture between 1910 and 1940. They argue that as long as agricultural wages were low, producers found it more profitable to operate the labor-intensive horse technology rather than the capital-intensive tractor.

All this evidence suggests that the degree of flexibility in agricultural production may be important for understanding the observed cross-country variation in sectoral capital-labor ratios and in sectoral labor productivities. In this view, agricultural production is labor-intensive in poor countries, which have relatively low wage to rental rate ratios. As countries develop and capital per worker increases, the flexible agricultural sector reacts strongly to the increase in the wage to rental rate ratio, substituting away from now relatively more expensive labor and into now relatively cheaper capital.[22] As a result, agricultural capital-labor ratios increase faster with development than non-agricultural ones.

[22] In a similar vein, Wingender (2015) traces variation in labor productivity between agriculture and non-agriculture to sectoral differences in the elasticity of substitution between skilled and unskilled labor.

5.2 Structural change out of agriculture in the United States

In this section, to complement the theoretical analysis, we quantitatively analyze an important historical episode of structural change: the movement of resources out of agriculture in the United States. We focus on a period for which estimates of sectoral substitution elasticities are available, namely 1960 to 2010.[23] At the beginning of this period, agriculture already was a much smaller sector than e.g. in the 19th century, but still accounted for almost 10 percent of hours worked, making it a non-negligible employer. At the end of this period, the fraction of employment in U.S. agriculture was less than two percent – a decline of 80%.

Figure 6 shows the fraction of U.S. employment in agriculture over the period 1960 to 2010. At the same time, it shows how the relative capital intensity of agriculture increased over this period: the fraction of U.S. capital used in agriculture starts at a similar level as that of labor, but then declines only by about half, to finish at about 5% of total capital. That is, while the fractions of capital and labor employed in agriculture start at similar levels in 1960, by 2010, the one of capital is more than twice as large as the one of labor. This also implies that the capital-labor ratio in agriculture grew at an average of 1.9% per year, or a third faster than that in the economy as a whole (1.4%).[24]

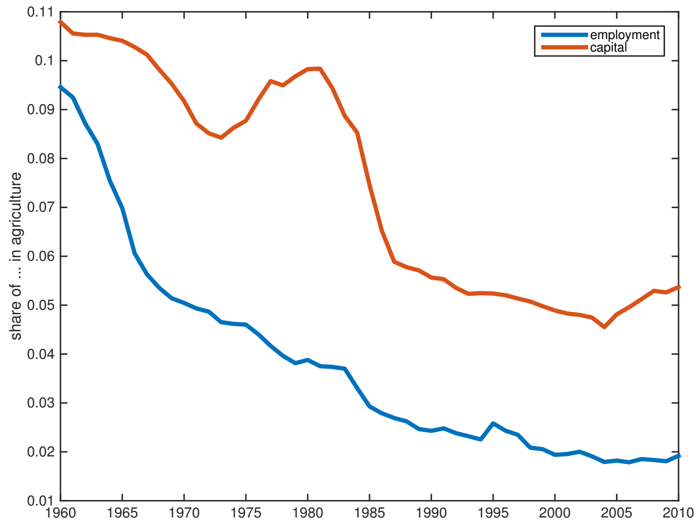
To investigate how well our simple model fits this episode, we calibrate it using, as far as possible, available estimates of preference and technology parameters. We calibrate the remaining parameters internally, using key information on the episode under consideration. We then evaluate how well the model fits the pattern of structural change just described, and conduct a decomposition that explores the contribution of the different drivers of structural change to these outcomes.

For realism and to do justice to the importance of dynamics when evaluating structural change over any given period of time, we slightly extend the model analyzed above. First, we assume that households maximize the discounted present value of log consumption. They choose saving optimally given current states and the path of technology. Secondly, we assume that only the non-agricultural good is used for investment, as this fits more closely with the data. Thirdly, because of its importance for the agricultural sector, we cannot abstract from the non-homotheticity of demand in doing so. We therefore assume that production of the

[23]These estimates (discussed below) as well as data used in this section are taken from Herrendorf, Herrington and Valentinyi (2015). Their data in turn are based on NIPA.

[24]While the measure of capital in Herrendorf, Herrington and Valentinyi (2015) includes land, this pattern is even more pronounced when land is excluded from the capital stock, with a growth rate of the capital-labor ratio in agriculture of 2.8%. In the analysis below, we use the data including land because the estimates of the sectoral substitution elasticities in Herrendorf, Herrington and Valentinyi (2015) – a key ingredient of our analysis – are obtained from these data.

Figure 6: The fractions of labor and capital used in agriculture, United States 1960-2010



Data source: Herrendorf, Herrington and Valentinyi (2014).

final consumption good can be described by the function

$$Y = \left[\gamma Y_n^{\frac{\varepsilon-1}{\varepsilon}} + (1-\gamma)(Y_a - \bar{a})^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$$

for $\varepsilon > 0$ and $Y = \min(\gamma(Y_a - \bar{a}), (1-\gamma)Y_n)$ for $\varepsilon = 0$, where $\bar{a}$ stands for agriculture and n for non-agriculture. A positive value of $\bar{a}$ implies that the income elasticity for agricultural value added is below that of non-agricultural value added. Preferences are homothetic if $\bar{a} = 0$. Finally, for simplicity, we abstract from factor-biased technical change, and assume that the level of Hicks-neutral productivity in each sector, D_s, grows at a constant rate g_{Ds}, which may differ across sectors.

5.2.1 Calibration and benchmark results

For preference parameters, we rely on estimates by Herrendorf, Rogerson and Valentinyi (2013, henceforth HRV). These authors estimate a CES demand system for value added from agriculture, manufacturing and services, using US data for the period 1947 to 2007. Translated into our context, their estimates imply $\varepsilon = 0$ and $\gamma = 0.99$. Furthermore, they estimate $\bar{a}$ to be positive. Since units for $\bar{a}$ are not directly comparable, we calibrate it internally and compare the size of $\bar{a}$ relative to consumption expenditure to HRV's estimates below. We also set the discount rate to 0.06.

For the substitution elasticities, we rely on estimates by Herrendorf, Herrington and Valentinyi (2015, henceforth HHV). These authors estimate CES production functions for the sectors agriculture, manufacturing and services and for the aggregate U.S. economy, using data for the period 1947 to 2010. They estimate $\sigma_a = 1.58$. For σ_n, we use their estimate for the aggregate economy, which is 0.84. (For comparison, the estimates for manufacturing and services are 0.8 and 0.75, respectively.) We take the initial aggregate capital-labor ratio from the data, at 12.2 1960 dollars per hour worked. We also set the depreciation rate to 0.05.

The remaining parameters are $\bar{a}, \alpha_a, \alpha_n, g_{Da}, g_{Dn}$, and the initial levels of sectoral productivity for 1960, D_{a0} and D_{n0}. We set D_{n0} to fit non-agricultural output in 1960 exactly, given observed sectoral factor inputs in that year. D_{a0} cannot be identified separately from $\bar{a}$, so we normalize it to 1. We set the non-homotheticity parameter $\bar{a}$ to match the allocation of labor across sectors in 1960. We then set the growth rate of non-agricultural productivity, g_{Dn}, to match the average growth rate of consumption per capita in the U.S. economy between 1960 and 2010, and that of agricultural productivity, g_{Da}, to match the allocation of labor in 2010, i.e. structural change in terms of labor over the period under consideration. Finally, we set α_n to match the aggregate capital income share in 1960, and α_a to match the allocation of capital across sectors in 1960.[25]

Resulting parameter values as well as data moments are shown in Table 2. The model fits targeted data moments exactly, so model moments are omitted from the table. To put it in perspective, the calibrated value for the non-homotheticity term $\bar{a}$ implies that "expenditure" on it accounts for 9.8% of spending on agricultural value added in 1960, compared to 8% as estimated by HRV. Note that the values of α_s do not directly translate into sectoral factor income shares. With a CES production function, these also depend on the factor inputs in each sector.

Most importantly, the estimates by HHV imply that the elasticity of substitution between capital and labor is much larger in agriculture compared to the rest of the economy. The resulting large difference in factor substitutability across sectors implies that the transition out of agriculture is a good testing ground for our model.

Since we target both the initial allocations of capital and labor across sectors and the final allocation of labor, the structural change outcome that is not restricted by the calibration is the final allocation of capital. This will be determined by the model and all parameters jointly. The implied model value for 2010 is 5.2%, virtually identical to the data value of 5.1%. The model can thus replicate almost exactly the structural transformation out of

[25]More precisely, all these parameters are calibrated jointly. However, we are mentioning here the most relevant target moment we use for each parameter.

Table 2: Calibration: parameters and model and data moments

Parameter	Value	Source/target	Target value
ε	0	HRV	
γ	0.99	HRV	
$\bar{a}$	0.217	L_{a0}/L_0	0.095
σ_a	1.58	HHV	
σ_n	0.84	HHV	
α_a	0.158	K_{a0}/K_0	0.108
α_n	0.446	R_0K_0/Y_0	0.325
D_{a0}	1	normalization	
D_{n0}	2.02	$Y_{n0}(K_{n0}, L_{n0})$	
g_{Da}	2.55%	L_{aT}/L_T	0.019
g_{Dn}	1.69%	g_c	2.4%

Note: time subscripts: 0 stands for 1960 and T for 2010. Sources for data/estimates: Herrendorf, Rogerson and Valentinyi (2013, HRV) and Herrendorf, Herrington and Valentinyi (2014, HHV).

agriculture in the U.S. in terms of the allocations of labor and of capital.[26] Figure 7 shows the model and data time paths for the shares of capital and labor employed in agriculture. It is clearly visible how in both model and data, the shares of capital and labor employed in agriculture drift apart.

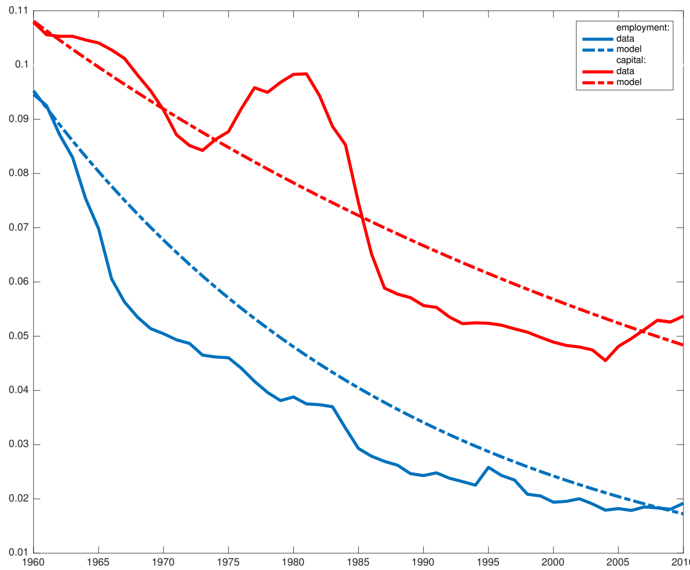
5.2.2 Decomposition: drivers of structural change

How important are the various potential drivers of structural change that have been proposed in the literature for the late stages of the transition out of agriculture in the United States? We answer this question by running a set of counterfactual experiments. In each experiment, we leave all drivers of structural change except for one in place. By comparing a set of economies with a missing driver of structural change, we can get a sense of their relative importance.

In a first experiment, we eliminate the effect of non-homothetic preferences by setting

[26] This is not entirely surprising given that these factor allocations enter the estimation in HHV, from which we take the values we use for substitution elasticities. At the same time, it is not self-evident that the model should do so well, since HHV do not use any cross-sectoral restrictions in their estimation, whereas these are front and center in our analysis.

Figure 7: The fractions of labor and capital used in agriculture, United States 1960-2010, and model economy



Data source: See Figure 6.

$\bar{a} = 0$. In a second scenario, we equate the productivity growth rate across sectors. We choose the common growth rate to keep the growth rate of aggregate consumption per capita the same as in the benchmark. In a third run, we equate the share parameters in the production function, α_s, across sectors. We pick the common value to maintain the initial labor allocation as in the data. Finally, to evaluate the importance of differences in factor substitutability, we equate σ_s across sectors. We set σ_s in each sector to the aggregate elasticity of substitution between capital and labor in the benchmark economy in 1960.[27] Results are shown in Table 3.

First, note again that the benchmark economy replicates almost exactly the experience of structural change out of agriculture to be found in the data: the fraction of U.S. labor employed in agriculture declined by 80%, while that of capital declined substantially less, by 50%. Looking across the rows recording results for the different experiments, it is clear that differences in σ are essential for the difference in structural change in terms of capital and labor. As long as the σ's differ across sectors, so does structural change in terms of capital and labor. Only in the last row, where substitution elasticities are equated across sectors, do capital and labor leave agriculture at a similar rate. This also implies that the growth rate of the capital-labor ratio is equated across the two sectors in this scenario, in contrast

[27] Results are similar if we choose it to keep the consumption growth rate unchanged.

Table 3: Decomposition: importance of different drivers of structural change

	L_{a0}/L_0	L_{aT}/L_T	decline	K_{a0}/K_0	K_{aT}/K_T	decline
data	9.5	1.9	80.0	10.8	5.1	52.8
benchmark	9.5	1.9	80.0	10.8	5.1	53.0
$\bar{a} = 0$	2.1	0.9	57.7	2.4	2.4	2.4
equal g_{Ds}	10.6	3.1	70.3	12.0	8.2	32.0
equal α_s	9.5	2.1	78.3	24.5	10.5	57.0
equal σ_s	15.4	6.5	57.6	4.6	1.8	60.5

Note: All figures are percentages. Line 4: $g_D = 1.7\%$. Line 5: $\alpha = 0.178$. Line 6: $\sigma = 0.91$.

to the data.

How much each channel matters for the amount of structural change that occurred is a question that requires a subtle answer. From the $\bar{a} = 0-$row, it is clear that the non-homotheticity is essential for replicating the large observed share of employment of capital and labor in agriculture in 1960. This is because the very small share of agricultural value added at the end of the sample implies such a low value for $1-\gamma$ that it is near impossible to obtain agricultural employment shares of 10% and above with homothetic preferences. Then, given the small size of the agricultural sector in 1960 implied by homothetic preferences, it is clear that little scope for absolute decline of the sector remains. At the same time, the relative decline in employment in agriculture is still substantial even when preferences are homothetic. Another result to notice is that the share of capital in agriculture hardly declines in the counterfactual scenario. The reason is that the low income elasticity of agricultural value added under non-homothetic preferences tends to reduce the relative price of agricultural output as the economy grows. In the benchmark calibration, this corresponds to an additional upward shift of the LM curve. In the counterfactual, this is removed. As a result, the factor rebalancing effect is relatively more important in the counterfactual, and structural change is even more asymmetric than in the benchmark.

The lower part of the table shows results for counterfactual scenarios where the supply-side channels are disabled one by one. It is clear that with equal sectoral productivity growth rates, structural change is slower. Nonetheless, there is still a large amount of input reallocation driven by the non-homotheticity, the increase in the aggregate capital-labor ratio, and the higher flexibility in agriculture. This is in constrast to earlier studies using Cobb-Douglas technologies, like e.g. Dennis and Iscan (2009), which have attributed a large fraction of the post-war structural change out of agriculture to differential sectoral productivity growth.[28]

Differences in the parameter governing sectoral factor intensity, α, hardly matter. Line 5 of the table shows results for equating α across sectors at a value chosen to keep the initial labor allocation as in the benchmark.[29] This implies a strong reduction in α_n, and accordingly, a strong reduction in the share of capital used outside agriculture. Structural change, however, is essentially unaffected.

Differences in the flexibility of the two sectors matter for the shape of structural change in terms of reallocation of capital and labor, as already remarked. They also are an important contributor to the measure of structural change that typically receives most attention, namely the reduction in the employment share of agriculture. With common σ, this declines by only a bit more than half, compared to 80% in the benchmark. At the same time, the fraction of capital employed in agriculture declines slightly faster when σ is common. The mechanism is clear: with common σ, reallocation of capital and labor is similar. When agriculture is more flexible, the optimal reaction is for the capital-labor ratio in agriculture to increase.

Comparing the four channels, it is clear that non-homothetic preferences are essential for matching the level of agricultural employment in 1960. The decline in the relative importance of agricultural employment is mostly driven by differences in capital-labor substitutability across sectors and by non-homothetic preferences. Finally, asymmetries in the movement of different factors out of agricultural are almost exclusively due to differences in factor substitutability across sectors.

6 Conclusions

We have developed a two-sector model where differences in the sectoral elasticity of substitution between capital and labor lead to a process of structural change. The mechanism is simple. As the wage to rental rate ratio changes, the more flexible sector – the sector with a higher elasticity of substitution between capital and labor – is in a better position to take

[28] Intuitively, productivity growth differences are less important when the fast-growing sector is also the one with high substitution elasticity. This is because in the absence of income and relative price effects, capital accumulation at the aggregate level leads to faster growth in the sector with higher substitution elasticity, in a similar way that faster productivity growth does. See Klump and De La Grandville (2000) for the relationship between the elasticity of substitution and growth in a one-sector model.

[29] We present these results for ease of interpretation; choosing the common value of α according to other criteria, e.g. to keep the aggregate capital income share in the initial period unchanged, leads to extreme changes in the initial allocation, but similar results in terms of structural change.

advantage of these changes than the less flexible one. As a result, sectoral capital-labor ratios grow at different rates, and the fractions of aggregate capital and labor allocated to a sector change by different amounts. It is even possible for the fraction of aggregate capital allocated to a sector to increase, while the fraction of labor declines. This is in contrast to the drivers of structural change emphasized in previous work, like differences in sectoral rates of total factor productivity growth (as in NP) or differences in capital intensity (as in AG), which tend to affect allocations of both factors in similar ways. Similarly to AG, structural change in our model ceases in the limit, and the economy eventually reaches a constant growth path where the fractions of employment and capital in both sectors are positive and constant.

Estimates of sectoral substitution elasticities show that these differ across sectors, and that these differences can be large (see e.g. Herrendorf, Herrington and Valentinyi 2015). In the last section, we have shown that these differences matter for thinking about structural change out of agricultural in the United States. They probably also matter for explaining differences in capital intensity and labor productivity of agriculture across countries. Our model also holds promise for understanding changes in the factor income share, which have recently been stressed by Elsby, Hobijn and Sahin (2013) and Karabarbounis and Neiman (2014), among others. Both issues are promising areas for future work. In our companion paper (Alvarez-Cuadrado, Long and Poschke 2015), we address the latter and analyze changes in sectoral and aggregate labor income shares in the U.S. from 1960 to 2005, a period of intense structural change from manufacturing to services.

Finally, although there have been several attempts to estimate the aggregate elasticity of substitution, our analysis suggests that, in general, this aggregate elasticity is a time-varying combination of deeper structural parameters, particularly of the sectoral elasticities of substitution. In this sense, another natural extension of this project would be to pursue the estimation of these elasticities at different levels of aggregation, along the lines of Herrendorf et al. (2015). Through the link between sectoral and aggregate elasticities of substitution given by equation (14), this exercise would also allow recovering an estimate of this aggregate elasticity. Furthermore, these sectoral elasticities could also be used to test some of the predictions of the model. In principle, existing datasets, such as the 35-industry KLEM developed by Dale W. Jorgenson and the EU-KLEMS gathered by the Groningen Growth and Development Center, provide the sectoral level data required for these estimations.

References

Acemoglu, D. and V. Guerrieri, (2008), "Capital Deepening and Nonbalanced Economic Growth", Journal of Political Economy, 116(3), 467-498.

Alvarez-Cuadrado, F., N.V. Long, and M. Poschke, (2015), "Capital-labor substitution, Structural Change and the Labor Income Share', IZA Discussion Paper No. 8941.

Alvarez-Cuadrado, F. and M. Poschke, (2011), "Structural Change Out of Agriculture: Labor Push versus Labor Pull" American Economic Journal: Macroeconomics, 3, 127-158.

Antras, P., (2004), "Is the U.S. Aggregate Production Function Cobb-Douglas? New Estimates of the Elasticity of Substitution" In: Contributions to Macroeconomics, vol 4. Berkeley Electronic Press.

Antras, P. and R. J. Caballero, (2009), "Trade and Capital Flows: A Financial Frictions Perspective," Journal of Political Economy, 117(4), 701-744.

Baumol, William J.(1967), "Macroeconomics of Unbalanced Growth: The Anatomy of Urban Crisis", American Economic Review, LVII , 415-426.

Bentolila, S., Saint-Paul, G., (2003), "Explaining Movements in the Labor Share," In: Contributions to Macroeconomics, vol. 3. Berkeley Electronic Press.

Blanchard, O.J., (1997), "The Medium Run", Brookings Papers on Economic Activity, 1997 –2, 89-158.

Boppart, T., (2014), "Structural change and the Kaldor facts in a growth model with relative price effects and non-Gorman preferences", Econometrica 82(6), 2167-2196.

Buera, F. and J.P. Kaboski (2012), "The Rise of the Service Economy," American Economic Review.

Buera, F., J.P. Kaboski and R. Rogerson (2015), "Skill-biased Structural Change", NBER Working Paper 21165.

Caballero, R. J. and M.L. Hammour, (1998), "Jobless growth: appropriability, factor substitution, and unemployment," Carnegie-Rochester Conference Series on Public Policy, 48(1), 51-94.

Caselli, F. and J. Coleman, (2001), "The U.S. Structural Transformation and Regional Convergence: A Reinterpretation", Journal of Political Economy 109, 584–616.

Chanda, A. and C. Dalgaard, (2008), "Dual Economies and International Total Factor Productivity Differences: Channelling the Impact from Institutions, Trade, and Geography," Economica, 75(300), 629-661.

Clark, C. (1940), The conditions of economic progress, 3rd edn, Macmillan, London.

Crego, A., D.F. Larson, R. Butzer, and Y. Mundlak, Yair (2000) "A Cross-Country

Database for Sector Investment and Capital," World Bank Economic Review, World Bank Group, vol. 14(2), pages 371-91.

Dennis, B. N. and Iscan, T. B., 2009. "Engel versus Baumol: Accounting for structural change using two centuries of U.S. data," Explorations in Economic History, vol. 46(2), pages 186-202.

Duarte, M. and Restuccia, D. (2010), "The Role of the Structural Transformation in Aggregate Productivity", Quarterly Journal of Economics 125(1).

Echevarria, C., (2007), "Changes in Sectoral Composition Associated with Economic-Growth", International Economic Review, XXXVIII, 431-452.

Elsby, M. W. L., B. Hobijn, and A. Sahin, (2013), "The Decline of the U.S. Labor Share", Brookings Papers on Economic Activity .

Foellmi R. and J. Zweimüller, (2008), "Structural Change, Engel's Consumption Cycles and Kaldor's Facts of Economic Growth", Journal of Monetary Economics 55(7), 1317-1328,

Gollin, D., D. Lagakos and M. E Waugh, (2014) "The Agricultural Productivity Gap", Quarterly Journal of Economics 129(2), 939-993.

Gollin, D., S. Parente and R. Rogerson, (2007), "The Food Problem and the Evolution of International Income Levels", Journal of Monetary Economics 54(4): 1230-1255.

Herrendorf, B., Herrington, C. and Valentinyi, A. (2015), "Sectoral technology and structural transformation", American Economic Journal: Macroeconomics 7, 1-31.

Herrendorf, B., Rogerson, R.. and Valentinyi, A. (2013), "Two Perspectives on Preferences and Structural Transformation", American Economic Review 103, 2752–2789.

Hicks, J. R., (1932), The Theory of Wages, 1st edition, London.

Jorgenson, D. (2007), "35 Sector KLEM",
http://hdl.handle.net/1902.1/10684 UNF:3:TqM00zRqsatX2q/teT253Q== V1.

Jensen, B.S. (2003), "Walrasian General Equilibrium Allocations and Dynamics in Two-Sector Growth Models", German Economic Review, 4, 53-87.

Jones, R.W., (1965), "The structure of simple general equilibrium models", Journal of Political Economy 73, 557-572.

Kaldor, N., (1963) "Capital Accumulation and Economic Growth", in Friedrich A. Lutz and Douglas C. Hague, eds., Proceedings of a Conference Held by the International Economics Association, London, Macmillan.

Karabarbounis, L. and B. Neiman. (2014), 'The Global Decline of the Labor Share', Quarterly Journal of Economics 129(1), 61-103.

Klump, R. and O. De La Grandville, (2000), "Economic Growth and the Elasticity of

Substitution: Two Theorems and Some Suggestions," American Economic Review, 90:1, pp. 282-291.

Kongsamut, P., Rebelo, S. and Xie, D. (2001), "Beyond Balanced Growth", Review of Economic Studies 68(4), 869–882.

Krusell, P., L. E. Ohanian, J. Rios-Rull, G. L. Violante (2000), "Capital-Skill Complementarity and Inequality: A Macroeconomic Analysis," Econometrica, 68(5), 1029-1054.

Kuznets, S. (1966), Modern economic growth, New Haven, CT: Yale University Press.

Larson, D. F,., R. Butzer, Y. Mundlak and A. Crego, Al, (2000), "A Cross-Country Database for Sector Investment and Capital," World Bank Economic Review, 14(2), 371-91.

Laitner, J. (2000), "Structural change and economic growth", Review of Economic Studies, 67, 545–561.

Leon-Ledesma, M., P. McAdam and A. Willman, (2010) "Identifying the Elasticity of Substitution with Biased Technical Change", American Economic Review 100(4), 1330-1357.

Leon-Ledesma, M., P. McAdam and A. Willman, (2015) "Production Technology Estimates and Balanced Growth", Oxford Bulletin of Economics and Statistics 77(1), 40-65.

Manuelli, R. and A. Seshadri, (2014) "Frictionless Technology Diffusion: The Case of Tractors," American Economic Review 104(4), 1368-91.

Matsuyama, K. (1992), "Agricultural productivity, comparative advantage, and economic growth", Journal of Economic Theory 58(2), 317–334.

Miyagiwa, K. and C. Papageorgiou, (2007), "Endogenous Aggregate Elasticity of Substitution", Journal of Economic Dynamics and Control, 31, 2899-2919.

Mundlak, Y., (2000), Agriculture and Economic Growth: Theory and Measurement, Cambridge, Massachusetts: Harvard University Press.

Ngai, R. and C. Pissarides, (2007), "Structural Change in a Multi-Sector Model of Growth", American Economic Review 97(1), 429–443.

Oberfield, E. and D. Raval, (2014), "Micro Data and Macro Technology", mimeo, Princeton University.

Pitchford, J. D. (1960), "Growth and the Elasticity of Substitution", Economic Record 36, 491-503

Reshef, A. (2013), "Technological Change Biased Towards the Unskilled in Services? An Empirical Investigation", Review of Economic Dynamics, 16, 312-31.

Restuccia, D., Yang, D. and Zhu, X. (2008), "Agriculture and Aggregate Productivity: A Quantitative Cross-Country Analysis", Journal of Monetary Economics 55(2), 234–50.

Rosenzweig, M., (1998), "Labor Markets in Low-Income Countries," in Hollis Chenery and T. N. Srinivasan, eds., Handbook of Development Economics, Vol.1, Part II, Chapter 15, 713-762, New York: North-Holland.

Takahashi, H. (2014), "Unbalanced Growth in A Neoclassical Two-sector Optimal Growth Model with Sector Specific Technical Progress: Baumol's Unbalanced Growth Revisited", mimeo, Meiji Gakuin University

Takayama, A., (1985), Mathematical Economics, 2nd Edition, Cambridge University Press.

Valentinyi, A., and B. Herrendorf (2008), "Measuring factor income shares at the sectoral level", Review of Economic Dynamics 11, 820-835.

Ventura, J., (1997), "Growth and interdependence", Quarterly Journal of Economics 112, 57-84.

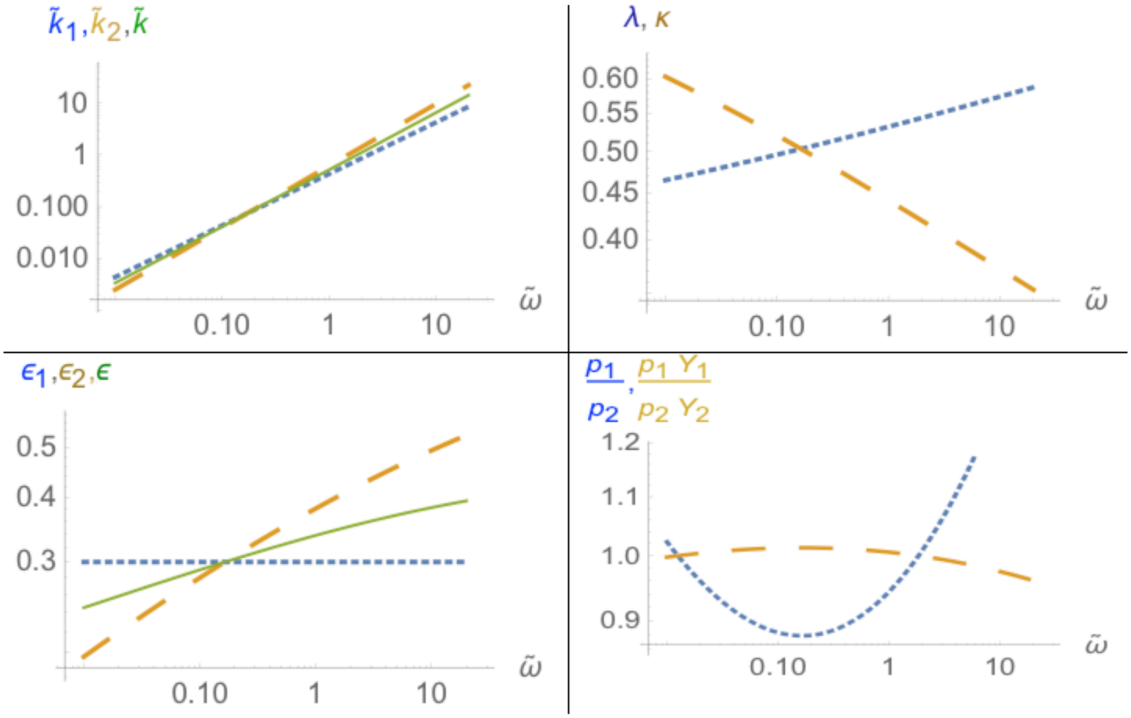
Wingender, A. M., (2015). "Skill complementarity and the dual economy," European Economic Review, Elsevier, 74(C), 269-285.

Zuleta, H, and A.T. Young, (2013) "Labor shares in a model of induced innovation," Structural Change and Economic Dynamics, 24(C), pages 112-122.

Appendix

A Additional figures

Figure 8: Model simulations, different values of σ_s (b)



Notes: Parameter settings as in Figure 4, except that $\varepsilon = 1.1$.

B Additional results and proofs

Proof of Proposition 1. Combining (7) with (9) we reach,

$$\kappa = (1-\kappa)^{1/\sigma_2}\left(\frac{\gamma}{1-\gamma}\right)\left[(1-\alpha)(1-\lambda)^{\frac{\sigma_2-1}{\sigma_2}}\left(\frac{k}{M_2}\right)^{\frac{1-\sigma_2}{\sigma_2}}+\alpha(1-\kappa)^{\frac{\sigma_2-1}{\sigma_2}}\right] \tag{24}$$

Re-arranging (CC) we reach,

$$(k/M_2)^{\frac{1-\sigma_2}{\sigma_2}} = \frac{(1-\lambda)^{\frac{1}{\sigma_2}}}{\lambda}\frac{\kappa}{(1-\kappa)^{1/\sigma_2}} \tag{25}$$

Equation (25) yields the implicit function

$$\lambda = \Lambda(\kappa, k/M_2)$$

with

$$\frac{\partial\Lambda(\kappa, k, M_2)}{\partial\kappa} > 0$$

and

$$\text{Sign}\frac{\partial\Lambda(\kappa, k/M_2)}{\partial(k/M_2)} = \text{Sign}(\sigma_2 - 1)$$

To simplify notation, let $\widetilde{\lambda}$ stand for $\Lambda(\kappa, k/M_2)$. Re-arranging (24)

$$\kappa-(1-\kappa)^{1/\sigma_2}\left(\frac{\gamma}{1-\gamma}\right)\alpha(1-\kappa)^{\frac{\sigma_2-1}{\sigma_2}} = (1-\kappa)^{1/\sigma_2}\left(\frac{\gamma}{1-\gamma}\right)(1-\alpha)(1-\widetilde{\lambda})^{\frac{\sigma_2-1}{\sigma_2}}(k/M_2)^{\frac{1-\sigma_2}{\sigma_2}} \tag{26}$$

Substitute (25) into (26)

$$\frac{\kappa}{(1-\kappa)^{1/\sigma_2}} - \left(\frac{\gamma}{1-\gamma}\right)\alpha(1-\kappa)^{\frac{\sigma_2-1}{\sigma_2}} = \left(\frac{\gamma}{1-\gamma}\right)(1-\alpha)(1-\widetilde{\lambda})^{\frac{\sigma_2-1}{\sigma_2}}\frac{(1-\widetilde{\lambda})^{\frac{1}{\sigma_2}}}{\widetilde{\lambda}}\left(\frac{\kappa}{(1-\kappa)^{1/\sigma_2}}\right)$$

or

$$1-\left(\frac{\gamma}{1-\gamma}\right)\alpha(1-\kappa)^{\frac{\sigma_2-1}{\sigma_2}}\left(\frac{(1-\kappa)^{1/\sigma_2}}{\kappa}\right) = \left(\frac{\gamma}{1-\gamma}\right)(1-\alpha)\left(\frac{1-\widetilde{\lambda}}{\widetilde{\lambda}}\right)$$

or

$$1-\left(\frac{\gamma}{1-\gamma}\right)\alpha\left(\frac{1-\kappa}{\kappa}\right) = \left(\frac{\gamma}{1-\gamma}\right)(1-\alpha)\left(\frac{1-\widetilde{\lambda}}{\widetilde{\lambda}}\right)$$

or

$$1-\left(\frac{\gamma}{1-\gamma}\right)\alpha\left(\frac{1}{\kappa}-1\right) = \left(\frac{\gamma}{1-\gamma}\right)(1-\alpha)\left(\frac{1}{\widetilde{\lambda}}-1\right)$$

Thus

$$\frac{\alpha\gamma}{\kappa} = \frac{\widetilde{\lambda}-(1-\gamma)}{\widetilde{\lambda}(1-\gamma)}$$

or equivalently,

$$\widetilde{\lambda} = \frac{\gamma\,(1-\alpha)\,\kappa}{(\kappa - \alpha\gamma)} \equiv \lambda(\kappa,\gamma,\alpha) \tag{27}$$

These two equations equations imply that, *in equilibrium*, $\lambda > (1-\gamma)$ and $\kappa > \alpha\gamma$ in order to satisfy $\kappa > 0$ and $\lambda > 0$ respectively. Also, since $\lambda \leq 1$ and $\kappa \leq 1$, they imply that

$$\kappa > \underline{\kappa} \equiv \frac{\alpha\gamma}{(1-\gamma)+\alpha\gamma} < 1$$

and

$$\lambda > \underline{\lambda} \equiv \frac{\gamma(1-\alpha)}{1-\alpha\gamma} < 1$$

Now, since

$$(\kappa - \alpha\gamma)\,\widetilde{\lambda} - \gamma\,(1-\alpha)\,\kappa = 0$$

let us define

$$\Omega(\kappa, k/M_2) \equiv (\kappa - \alpha\gamma)\,\Lambda(\kappa, k/M_2) - \gamma\,(1-\alpha)\,\kappa = 0$$

This equation yields κ as an implicit function of k and M_2, which we denote as $\kappa = \kappa^*(k/M_2)$. Then

$$\frac{d\kappa^*}{d(k/M_2)} = -\frac{\dfrac{\partial\Omega}{\partial(k/M_2)}}{\dfrac{\partial\Omega}{\partial\kappa}}$$

where

$$\frac{\partial\Omega}{\partial\kappa} = [\Lambda(\kappa, k/M_2) - \gamma\,(1-\alpha)] + (\kappa-\alpha\gamma)\,\frac{\partial\Lambda(\kappa, k/M_2)}{\partial\kappa} > 0$$

and

$$\frac{\partial\Omega}{\partial(k/M_2)} = (\kappa-\alpha\gamma)\,\frac{\partial\Lambda(\kappa, k/M_2)}{\partial(k/M_2)} > 0 \text{ iff } \sigma_2 > 1$$

Therefore $\dfrac{d\kappa^*}{d(k/M_2)} < 0$ iff $\sigma_2 > 1$. Finally,

$$\lambda^*(k/M_2) = \Lambda(\kappa^*(k/M_2), k/M_2) = \frac{\gamma\,(1-\alpha)\,\kappa^*(k/M_2)}{\kappa^*(k/M_2)-\alpha\gamma} \tag{28}$$

Thus

$$\frac{d\lambda^*}{d(k/M_2)} = -\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\lambda^*}{\kappa^*}\right)^2 \frac{d\kappa^*}{d(k/M_2)} \tag{29}$$

This equation shows that $\lambda^*(k/M_2)$ and $\kappa^*(k/M_2)$ always move in opposite directions as k/M_2 increases (even though $\partial\Lambda(\kappa, k/M_2)/\partial\kappa$, i.e., the slope of the contract curve, for a given k/M_2, is always positive).

Combining (25) and (27) and taking logs we reach,

$$\frac{1-\sigma_2}{\sigma_2}\ln\left(\frac{k}{M_2}\right)=\frac{1}{\sigma_2}\ln(1-\lambda(\kappa,\alpha,\gamma))-\ln\lambda(\kappa,\alpha,\gamma)+\ln\kappa-\frac{1}{\sigma_2}\ln(1-\kappa)$$

This relationship is monotone decreasing (iff $\sigma_2>1$): an increase in $\dfrac{k}{M_2}$ leads to a fall in κ :

$$\left(\frac{1-\sigma_2}{\sigma_2}\right)\left(\frac{dk}{k}-\frac{dM_2}{M_2}\right)=G\left(\kappa\right)d\kappa \tag{30}$$

where

$$G(\kappa)\equiv\left[\frac{1}{\sigma_2\left(1-\lambda(\kappa,\alpha,\gamma)\right)}+\frac{1}{\lambda(\kappa,\alpha,\gamma)}\right]\left(\frac{\lambda(\kappa,\alpha,\gamma)}{\kappa}\right)^2\left(\frac{\alpha}{1-\alpha}\right)+\left[\frac{1}{\kappa}+\frac{1}{\sigma_2(1-\kappa)}\right]>0$$

where $\lambda(\kappa,\alpha,\gamma)$ is given by (27). Combining equations (29) and (30) we reach the results summarized in this proposition. ■

Proof of Proposition 2. Let $\sigma_1=\sigma_2=1$ and $\alpha_2>\alpha_1$ so that sector 1 is labor-intensive, as in AG. Under these assumptions the CC curve simplifies to

$$CC(\kappa,\lambda)\equiv\frac{(1-\alpha_1)}{(1-\alpha_2)}\frac{\alpha_2}{\alpha_1}\frac{\kappa}{(1-\kappa)}\frac{(1-\lambda)}{\lambda}=1, \tag{31}$$

which implicitly defines $\kappa=\Gamma\left(\lambda\right)$. Then the LM curve reduces to

$$\lambda-\left[1+\frac{(1-\gamma)}{\gamma}\frac{(1-\alpha_2)}{(1-\alpha_1)}\left(k^{(\alpha_1-\alpha_2)}\frac{M_1^{(1-\alpha_1)}}{M_2^{(1-\alpha_2)}}\frac{\Gamma\left(\lambda\right)^{\alpha_1}}{(1-\Gamma\left(\lambda\right))^{\alpha_2}}\frac{\lambda^{(1-\alpha_1)}}{(1-\lambda)^{(1-\alpha_2)}}\right)^{\frac{1-\varepsilon}{\varepsilon}}\right]^{-1}=0.$$

Differentiating this expression, treating κ as a function of λ, we obtain the results summarized in Proposition 2. ■

Proof of Proposition 3. Let $\sigma_1=\sigma_2=1$ and $\alpha_1=\alpha_2=\alpha$, as in NP. Under these assumptions the CC curve simplifies to

$$CC(\kappa,\lambda)\equiv\frac{\kappa}{(1-\kappa)}\frac{(1-\lambda)}{\lambda}=1, \tag{32}$$

which implies that $\lambda=\kappa$. Then the LM curve reduces to

$$\lambda=\left[1+\left(\frac{1-\gamma}{\gamma}\right)^{\varepsilon}\left(\frac{M_2}{M_1}\right)^{(1-\alpha)(\varepsilon-1)}\right]^{-1}. \tag{33}$$

The results in this Proposition follow from the log-differentiation of equation (33).

Proof of Lemma 3.

Denote the intensive form of the production function in each sector by $f_s(k_s)$. Using this and denoting the wage-rental ratio by ω, combine (7) and (8) to obtain

$$\omega = \frac{w}{R} = \frac{MPL_s}{MPK_s} = \frac{f_s(k_s;M_s) - k_s f_s'(k_s;M_s)}{f_s'(k_s)} = \frac{f_s}{f_s'} - k_s,$$

i.e.,

$$\frac{f_s'}{f_s} = \frac{1}{\omega + k_s}. \tag{34}$$

Hence

$$\frac{\mathrm{d}\omega}{\mathrm{d}k_s} = -\frac{f_s f_s''}{(f_s')^2} > 0. \tag{35}$$

Then rewrite (7), the condition prescribing that the marginal value product of capital is equated across sectors, as $p_1 f_1'(k_1(\omega), M_1) = p_2 f_2'(k_2(\omega), M_2)$, where the function arguments make explicit that the optimal capital-labor ratio in each sector depends on the input price ratio ω. Rearranging and taking logarithms on both sides, we obtain

$$\ln\frac{p_1}{p_2} = \ln f_2'(k_2(\omega), M_2) - \ln f_1'(k_1(\omega), M_1).$$

Thus

$$\begin{aligned}
\frac{\mathrm{d}\ln(p_1/p_2)}{\mathrm{d}\omega} &= \frac{1}{f_2'} f_2'' \frac{\mathrm{d}k_2}{\mathrm{d}\omega} - \frac{1}{f_1'} f_1'' \frac{\mathrm{d}k_1}{\mathrm{d}\omega} = -\frac{f_2'}{f_2} + \frac{f_1'}{f_1} \\
&= -\frac{1}{\omega + k_2} + \frac{1}{\omega + k_1} \\
&= \frac{k_2 - k_1}{(\omega + k_2)(\omega + k_1)} \lesseqqgtr 0 \text{ for } \omega \lesseqqgtr \tilde{\omega},
\end{aligned}$$

where the second equality uses (35) and the third uses (34), and $\bar{\omega}$ denotes the level of the wage-rental ratio implied by $k = \bar{k}$. Given the negative relationship between ω and k, this implies that the relative price of sector 1 output declines in k for k below $\bar{k}$, increases in k for k above $\bar{k}$, and reaches a minimum at $\bar{k}$.

The remaining two claims follow from combining this result with (10). ■

Proposition 6 Increases in k. *Assume $\sigma_2 > \sigma_1$. Define the following three conditions:*

$$\left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right)\epsilon_1 > \left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right)\epsilon_2 \tag{P6.A}$$

$$\epsilon_1 - \epsilon_2 < \frac{\sigma_2 - \sigma_1}{\varepsilon - \sigma_2}(1 - \epsilon_1) \tag{P6.B}$$

$$\epsilon_1 - \epsilon_2 > \frac{\sigma_2 - \sigma_1}{\sigma_2 - \varepsilon}\epsilon_1 \tag{P6.C}$$

Then an increase in the aggregate capital-labor ratio k yields the following changes in λ and κ:

1. *If $\sigma_1, \sigma_2 < \varepsilon$, the relative marginal product effect dominates the relative price effect, and (LM) is upward-sloping in κ, λ-space.*

   (a) *If condition P6.A holds, an increase in k shifts (LM) up.*

      i. *If condition P6.B holds, the upward shift of (CC) at unchanged κ exceeds that of (LM). ($g(\lambda_{CC}) > g(\lambda_{LM})$.) Hence, λ increases and κ decreases.*

      ii. *If condition P6.B does not hold, the upward shift of (LM) at unchanged κ exceeds that of (CC). ($g(\lambda_{LM}) > g(\lambda_{CC})$.) Hence, λ and κ both increase.*

   (b) *If condition P6.A does not hold, an increase in k shifts (LM) down in κ, λ-space. Hence, λ and κ both decrease.*

2. *If $\sigma_1, \sigma_2 > \varepsilon$, the relative price effect dominates the relative marginal product effect, and (LM) is downward-sloping in κ, λ-space.*

   (a) *If condition P6.A holds, an increase in k shifts (LM) up.*

      i. *If condition P6.B does not hold, the upward shift of (CC) at unchanged κ exceeds that of (LM). ($g(\lambda_{CC}) > g(\lambda_{LM})$.) Hence, λ increases and κ decreases.*

      ii. *If condition P6.B holds, the upward shift of (LM) at unchanged κ exceeds that of (CC). ($g(\lambda_{LM}) > g(\lambda_{CC})$.) Hence, λ and κ both increase.*

   (b) *If condition P6.A does not hold, an increase in k shifts (LM) down in κ, λ-space.*

      i. *If condition P6.C holds, the leftward shift of (CC) at unchanged λ exceeds that of (LM). ($|g(\kappa_{CC})| > |g(\kappa_{LM})|$.) Hence, λ increases and κ decreases.*

      ii. *If condition P6.C does not hold, the leftward shift of (LM) at unchanged λ exceeds that of (CC). ($|g(\kappa_{LM})| > |g(\kappa_{CC})|$.) Hence, λ and κ both decrease.*

Proof of Proposition 6.

Determination of the condition distinguishing a. and b. This is simply Lemma 2 (Shifts in the LM curve).

Determination of the condition distinguishing a.i. and a.ii. Consider a situation where k changes by a proportion $g(k)$, and κ remains unchanged. Then (CC) requires a change in λ of

$$g(\lambda_{CC}) = \frac{\frac{1}{\sigma_1} - \frac{1}{\sigma_2}}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\frac{\lambda}{1-\lambda}} g(k).$$

For (LM) to hold, a change in λ of

$$g(\lambda_{LM}) = \frac{\left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right)\epsilon_1 - \left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right)\epsilon_2}{\frac{1}{\sigma_1}\epsilon_1 + \frac{1}{\sigma_2}\epsilon_2\frac{\lambda}{1-\lambda} + \frac{1}{\varepsilon}\left((1-\epsilon_1) + (1-\epsilon_2)\frac{\lambda}{1-\lambda}\right)} g(k)$$

is needed. Under the assumption that $\sigma_2/\sigma_1 > \epsilon_2/[(1-\frac{\sigma_1}{\varepsilon})\epsilon_1 + \frac{\sigma}{\varepsilon}\epsilon_2]$, this is positive. Comparing these two expressions implies that $g(\lambda_{CC}) > g(\lambda_{LM})$ if

$$\begin{aligned} &\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)\left(\frac{1}{\sigma_1}\epsilon_1 + \frac{1}{\sigma_2}\epsilon_2\frac{\lambda}{1-\lambda} + \frac{1}{\varepsilon}\left((1-\epsilon_1) + (1-\epsilon_2)\frac{\lambda}{1-\lambda}\right)\right) \\ &> \left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\frac{\lambda}{1-\lambda}\right)\left(\left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right)\epsilon_1 - \left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right)\epsilon_2\right). \end{aligned}$$

After some tedious algebra, this condition becomes

$$\frac{\epsilon_2 - \epsilon_1}{\sigma_1\sigma_2} - \frac{1-\epsilon_1}{\varepsilon\sigma_2} + \frac{1-\epsilon_2}{\varepsilon\sigma_1} > 0$$

or

$$\epsilon_1 - \epsilon_2 < \frac{(1-\epsilon_2)\,\sigma_2 - (1-\epsilon_1)\,\sigma_1}{\varepsilon}.$$

Defining $\vartheta \equiv \epsilon_1 - \epsilon_2$, substituting out ϵ_2 using ϑ and ϵ_1, and solving for ϑ then yields

$$\vartheta\,(\varepsilon - \sigma_2) < (1-\epsilon_1)\,(\sigma_2 - \sigma_1)$$

This comparison reveals that

$$g(\lambda_{CC}) > g(\lambda_{LM}) \quad \text{if} \quad \frac{\sigma_2 - \sigma_1}{\varepsilon - \sigma_2}(1-\epsilon_1) > \epsilon_1 - \epsilon_2 \quad \text{if } \sigma_2 < \varepsilon \tag{36}$$

$$g(\lambda_{CC}) > g(\lambda_{LM}) \quad \text{if} \quad \frac{\sigma_2 - \sigma_1}{\varepsilon - \sigma_2}(1-\epsilon_1) < \epsilon_1 - \epsilon_2 \quad \text{if } \sigma_2 > \varepsilon, \tag{37}$$

or condition P6.B in Proposition 6.

It is clear from the graphical analysis (see Figure 1) that for both (CC) and (LM) to hold when (CC) shifts up more than (LM), it is required that λ increases, but κ declines.

Analysis of case 1.b. Recall that in case 1., (LM) is upward-sloping and flatter than (CC), since for any κ, allocating the entire labor endowment to one of the two sectors is never a solution. As k increases, (LM) shifts down and (CC) shifts up. This implies that both κ and λ decline.

Determination of the condition distinguishing 2.b.i. and 2.b.ii. In this case, the change in λ depends on whether (LM) or (CC) shifts left more. To check this, consider a situation where k changes by a proportion $g(k)$, and λ remains unchanged. Then (CC) requires a change in κ of

$$g(\kappa_{CC}) = \frac{\frac{1}{\sigma_2} - \frac{1}{\sigma_1}}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\frac{\kappa}{1-\kappa}} g(k) < 0.$$

For (LM) to hold with unchanged λ, a change in κ of

$$g(\kappa_{LM}) = \frac{\left(\frac{1}{\varepsilon} - \frac{1}{\sigma_1}\right)\epsilon_1 - \left(\frac{1}{\varepsilon} - \frac{1}{\sigma_2}\right)\epsilon_2}{\left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right)\epsilon_1 + \left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right)\epsilon_2 \frac{\kappa}{1-\kappa}} g(k)$$

is needed. Under the assumptions that $\sigma_1, \sigma_2 > \varepsilon$ and that condition P6.A does not hold, this is negative. After some tedious algebra, comparing these two expressions reveals that

$$g(\kappa_{LM}) > g(\kappa_{CC}) \quad \text{if} \quad \epsilon_1 - \epsilon_2 > \frac{\sigma_2 - \sigma_1}{\sigma_2 - \varepsilon}\epsilon_1. \tag{P6.C'}$$

This is condition P6.C in Proposition 6. Since both changes in κ are negative, (CC) shifts left more under this condition. It is clear from the graphical analysis (see Figure 1) that in this case, for both (CC) and (LM) to hold, it is required that λ increases, but κ declines. ■

Proposition 7 Increases in M_1 and M_2. *Assume $\sigma_2 > \sigma_1$. Then a proportional increase in both M_1 and M_2 leads to the following changes in λ and κ:*

1. *$\sigma_1, \sigma_2 < \varepsilon$. As shown above, the relative marginal product effect dominates the relative price effect, and (LM) is upward-sloping in κ, λ-space in this case.*

   (a) *If condition P6.A holds, a proportional increase in M_1 and M_2 shifts (LM) down in κ, λ-space.*

      i. *If condition P6.B holds, the downward shift of (CC) at unchanged κ exceeds that of (LM). ($|g(\lambda_{CC})| > |g(\lambda_{LM})|$.) Hence, λ declines, but κ increases.*

      ii. *If condition P6.B does not hold, the downward shift of (LM) at unchanged κ exceeds that of (CC). ($|g(\lambda_{CC})| < |g(\lambda_{LM})|$.) Hence, both λ and κ decrease.*

   (b) *If condition P6.A does not hold, a proportional increase in M_1 and M_2 shifts (LM) up. Given the downward shift in (CC) and the positive slope of (LM), this implies that λ and κ both increase.*

2. *$\sigma_1, \sigma_2 > \varepsilon$. As shown above, the relative price effect dominates the relative marginal product effect, and (LM) is downward-sloping in this case.*

   (a) *If condition P6.A holds, a proportional increase in M_1 and M_2 shifts (LM) down in κ, λ-space.*

      i. *If condition P6.B holds, the downward shift of (LM) at unchanged κ exceeds that of (CC). ($|g(\lambda_{CC})| < |g(\lambda_{LM})|$.) Hence, both λ and κ decline.*

      ii. *If condition P6.B does not hold, the downward shift of (CC) at unchanged κ exceeds that of (LM). ($|g(\lambda_{LM})| < |g(\lambda_{CC})|$.) Hence, λ declines, but κ increases.*

   (b) *If condition P6.A does not hold, a proportional increase in M_1 and M_2 shifts (LM) up.*

      i. *If condition P6.C holds, the rightward shift of (CC) at unchanged λ exceeds that of (LM). ($g(\kappa_{CC}) > g(\kappa_{LM})$.) Hence, κ increases and λ declines.*

      ii. *If condition P6.C does not hold, the rightward shift of (LM) at unchanged λ exceeds that of (CC). ($g(\kappa_{LM}) > \mathrm{d}\kappa_{CC}$.) Hence, λ and κ both increase.*

Proof of Proposition 7.

Determination of the condition distinguishing a. and b. (LM) increases in M_1 and M_2 if

$$-\frac{\epsilon_1}{\sigma_1} + \frac{\epsilon_2}{\sigma_2} + \frac{\epsilon_1 - \epsilon_2}{\varepsilon} > 0.$$

Solving this for σ_2/σ_1 shows that this is the case if condition P6.A from Proposition 6 does not hold. Note that the result is the same in cases 1. and 2.

Since (LM) decreases in λ, λ needs to increase after a proportional increase in M_1 and M_2 if condition P6.A does not hold, and decrease if it holds. Hence, if condition P6.A holds, λ needs to decrease following a proportional increase in M_1 and M_2 for (LM) to hold.

Determination of the condition distinguishing a.i. and a.ii. Consider a situation where both M_1 and M_2 change by a proportion $g(M)$, and κ remains unchanged. Then (CC) requires a change in λ of

$$g(\lambda_{CC}) = \frac{\frac{1}{\sigma_2} - \frac{1}{\sigma_1}}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\frac{\lambda}{1-\lambda}} g(M) < 0.$$

For (LM) to hold, a change in λ of

$$g(\lambda_{LM}) = \frac{\left(\frac{1}{\sigma_2} - \frac{1}{\varepsilon}\right)\epsilon_2 - \left(\frac{1}{\sigma_1} - \frac{1}{\varepsilon}\right)\epsilon_1}{\frac{1}{\sigma_1}\epsilon_1 + \frac{1}{\sigma_2}\epsilon_2\frac{\lambda}{1-\lambda} + \frac{1}{\varepsilon}\left((1-\epsilon_1) + (1-\epsilon_2)\frac{\lambda}{1-\lambda}\right)} g(M)$$

is needed. Under the assumption that condition P6.A holds, this is negative. After some tedious algebra, comparing these two expressions reveals that

$$g(\lambda_{CC}) > g(\lambda_{LM}) \quad \text{if} \quad \frac{\sigma_2 - \sigma_1}{\varepsilon - \sigma_2}(1-\epsilon_1) < \epsilon_1 - \epsilon_2 \quad \text{if } \sigma_2 < \varepsilon \tag{38}$$

$$g(\lambda_{CC}) > g(\lambda_{LM}) \quad \text{if} \quad \frac{\sigma_2 - \sigma_1}{\varepsilon - \sigma_2}(1-\epsilon_1) > \epsilon_1 - \epsilon_2 \quad \text{if } \sigma_2 > \varepsilon \tag{39}$$

Since both changes in λ are negative, this implies that (CC) shifts down less under this condition. It is clear from the graphical analysis (see Figure 1) that in this case, for both (CC) and (LM) to hold, it is required that λ and κ both decline. If this condition does not hold, (CC) shifts down more, so that for both (CC) and (LM) to hold, it is required that λ declines but κ increases.

Analysis of case 1.b. In this case, (LM) is upward-sloping and shifts up. (CC) shifts down. This implies that both κ and λ increase.

Determination of the condition distinguishing b.i. and b.ii. Consider a situation where M_1 and M_2 change by a proportion $g(M)$, and λ remains unchanged. Then (CC) requires a change in κ of

$$g(\kappa_{CC}) = \frac{\frac{1}{\sigma_1} - \frac{1}{\sigma_2}}{\frac{1}{\sigma_1} + \frac{1}{\sigma_2}\frac{\kappa}{1-\kappa}} \mathrm{d}M > 0.$$

For (LM) to hold, a change in κ of

$$g(\kappa_{LM}) = \frac{\left(\frac{1}{\varepsilon} - \frac{1}{\sigma_1}\right)\epsilon_1 - \left(\frac{1}{\varepsilon} - \frac{1}{\sigma_2}\right)\epsilon_2}{\left(\frac{1}{\varepsilon} - \frac{1}{\sigma_1}\right)\epsilon_1 + \left(\frac{1}{\varepsilon} - \frac{1}{\sigma_2}\right)\epsilon_2\frac{\kappa}{1-\kappa}} g(M)$$

is needed. Under the assumption that condition P6.A does not hold, this is positive. After some tedious algebra, comparing these two expressions reveals that

$$g(\kappa_{LM}) > g(\kappa_{CC}) \quad \text{if} \quad \frac{\sigma_2 - \sigma_1}{\sigma_2 - \varepsilon}\epsilon_1 > \epsilon_1 - \epsilon_2,$$

or if condition P6.C does not hold. Since both changes in κ are positive, this implies that (LM) shifts right more under this condition. It is clear from the graphical analysis (see Figure 1) that in this case, for both (CC) and (LM) to hold, it is required that both κ and λ increase. ■

Proof of Lemma 4. This proof follows Jones (1965) and Miyagiwa and Papageourgiou (2007). The dual relationship between sectoral prices and input prices and factor endowments and sectoral outputs are given by,

$$C_1(w,R) \equiv \frac{L_1}{Y_1}w + \frac{K_1}{Y_1}R = p_1 \tag{40}$$

$$C_2(w,R) \equiv \frac{L_2}{Y_2}w + \frac{K_2}{Y_2}R = p_2 \tag{41}$$

$$Y_1 C_{1w} + Y_2 C_{2w} = L \tag{42}$$

$$Y_1 C_{1R} + Y_2 C_{2R} = K \tag{43}$$

where $C_i(w,R)$ is the unit cost function for sector $s = 1,2$ and C_{ij} are its partial derivatives with respect to each factor price $j = w, R$.

Differentiating the previous expressions we reach the following relationships,

$$\frac{wL_1}{p_1Y_1}\hat{w} + \frac{RK_1}{p_1Y_1}\hat{R} = (1-\epsilon_1)\hat{w} + \epsilon_1\hat{R} = \widehat{p_1} \tag{44}$$

$$\frac{wL_2}{p_2Y_2}\hat{w} + \frac{RK_2}{p_2Y_2}\hat{R} = (1-\epsilon_2)\hat{w} + \epsilon_2\hat{R} = \widehat{p_2} \tag{45}$$

$$\lambda\left(\hat{Y}_1+\widehat{C}_{1w}\right)+(1-\lambda)\left(\hat{Y}_2+\widehat{C}_{2w}\right)=\hat{L} \tag{46}$$

$$\kappa\left(\hat{Y}_1+\widehat{C}_{1R}\right)+(1-\kappa)\left(\hat{Y}_2+\widehat{C}_{2R}\right)=\hat{K} \tag{47}$$

where ϵ_s is the capital income share in sector s and we use the fact that sectoral production functions are homogeneous of degree one.

Subtracting (44) and (45),

$$(\epsilon_2-\epsilon_1)\left(\hat{w}-\hat{R}\right)=\widehat{p}_1-\widehat{p}_2 \tag{48}$$

Using the definition of the sector-specific elasticity of substitution, $\sigma_s \equiv \frac{C_s C_{swR}}{C_{sw}C_{sR}}$, since we can express the factor income shares as $\epsilon_s=\frac{rC_{sR}}{C_s}$ and $1-\epsilon_s=\frac{wC_{sw}}{C_s}$, we reach the following rates of change of partial derivatives of the unit cost functions,

$$\widehat{C}_{sw}=\frac{C_{sww}dw+C_{swr}dR}{C_{sw}}=\frac{-C_{swR}\frac{R}{w}dw+C_{swR}dR}{C_{sw}}=$$

$$-\frac{(C_{swR}R\hat{w}-C_{swR}dR)}{C_{sw}}=-\frac{C_{swR}R}{C_{sw}}\left(\hat{w}-\hat{R}\right)=-\frac{C_sC_{swR}}{C_{sw}C_{sR}}\frac{RC_{sR}}{C_s}\left(\hat{w}-\hat{R}\right)$$

where the second equality uses the fact that $C_{sw}\left(w,\hat{R}\right)$ is homogeneous of degree 0. As a result

$$\widehat{C}_{sw}=-\sigma_s\epsilon_s\left(\hat{w}-\hat{R}\right) \tag{49}$$

$$\widehat{C}_{sR}=\sigma_s\left(1-\epsilon_s\right)\left(\hat{w}-\hat{R}\right) \tag{50}$$

Replacing (49) and (50) in (46) and (47) and subtracting them we reach,

$$(\lambda-\kappa)\,(\hat{Y}_1-\hat{Y}_2)=\left(\hat{L}-\hat{K}\right)+\Theta\left(\hat{w}-\hat{R}\right) \tag{51}$$

where $\Theta\equiv\lambda\sigma_1\epsilon_1+(1-\lambda)\,\sigma_2\epsilon_2+\kappa\sigma_1\,(1-\epsilon_1)+(1-\kappa)\,\sigma_2\,(1-\epsilon_2)$.

Finally, we use (10) to reach

$$\hat{Y}_1-\hat{Y}_2=-\varepsilon\left(\widehat{p}_1-\widehat{p}_2\right) \tag{52}$$

Since the aggregate elasticity of substitution is defined as $\sigma\equiv-\frac{\left(\hat{L}-\hat{K}\right)}{\left(\hat{w}-\hat{R}\right)}$ we combine (48) and (51) in (52) to reach (14).

Proof of Proposition 4. Using (17) we can rewrite the CC curve as,

$$\frac{(1-\kappa)^{\frac{1}{\sigma_2}}}{\kappa}\frac{\lambda}{(1-\lambda)^{\frac{1}{\sigma_2}}}=\chi^{\frac{\sigma_2-1}{\sigma_2}}$$

which, after using (28) to replace λ, becomes

$$\chi=(\gamma(1-\alpha))^{\frac{\sigma_2}{\sigma_2-1}}\frac{(1-\kappa)^{\frac{1}{\sigma_2-1}}}{(\kappa-\alpha\gamma)\left(\kappa\left(1-\gamma(1-\alpha)\right)-\alpha\gamma\right)^{\frac{1}{\sigma_2-1}}}$$

where $\chi'(\kappa)<0$ (resp. $\chi'(\kappa)>0$) for all $\kappa\in(\underline{\kappa},1)$ if $\sigma_2>1$ (resp. $\sigma_2<1$). Furthermore, it is worth noticing that when $\sigma_2>1$, $\chi(1)=0$ and $\lim_{\kappa\to\underline{\kappa}}\chi(\kappa)=\infty$, and when $\sigma_2<1$, $\chi(\underline{\kappa})=0$ and $\lim_{\kappa\to 1}\chi(\kappa)=\infty$.

Given (17) and $M_1=M_2$,the rate of change of the normalized capital stock is,

$$\widehat{\chi}=\widehat{K}-\widehat{M_1}-\widehat{L}=v\frac{Y}{K}-\delta-m-n$$

Since $\varepsilon=1$ and sector 1 has the Cobb-Douglas technology while sector 2 has the CES technology, the agregate output-capital ratio is given by

$$\frac{Y}{K}=\left[(\lambda\chi^{-1})^{1-\alpha}(\kappa)^{\alpha}\right]^{\gamma}\left[(1-\alpha)\left((1-\lambda)\chi^{-1}\right)^{\frac{\sigma_2-1}{\sigma_2}}+\alpha((1-\kappa))^{\frac{\sigma_2-1}{\sigma_2}}\right]^{\frac{(1-\gamma)\sigma_2}{\sigma_2-1}}$$

which can be expressed using (**??**), (28), and (19) as

$$\frac{Y}{K}=D\pi(\kappa)\tag{53}$$

where D and $\pi(\kappa)$ are defined by (21) and (22) respectively. Notice that when $\sigma_2>1$ (respectively, $\gamma<\sigma_2<1$), $\pi(\kappa)$ is an increasing resp. decreasing) function defined over the interval $[\underline{\kappa},1]$, with $\pi(\underline{\kappa})=0$ (resp. $\pi(\kappa)=\infty$) and $\lim_{\kappa\to 1}\pi(\kappa)=\infty$ (resp. $\pi(1)=0$).

Therefore

$$\widehat{\chi}=vD\pi(\kappa)-(\delta+m+n)\tag{54}$$

Finally log-differentiating (19)

$$\widehat{\chi}=-H(\kappa)\dot{\kappa}\tag{55}$$

where $H(\kappa)$ is defined by

$$H(\kappa)\equiv-\left(\frac{1}{\sigma_2-1}\right)\frac{1}{1-\kappa}-\frac{(\sigma_2-1)}{(\kappa-\alpha\gamma)(\sigma_2-1)}-\left(\frac{1}{\sigma_2-1}\right)\frac{1-\gamma(1-\alpha)}{\kappa(1-\gamma(1-\alpha))-\alpha\gamma}$$

Then

$$H(\kappa) = -\left(\frac{1}{\sigma_2 - 1}\right)\left[\frac{1}{1-\kappa} + \frac{\sigma_2 - 1}{(\kappa - \alpha\gamma)} + \frac{1-\gamma(1-\alpha)}{\kappa(1-\gamma(1-\alpha)) - \alpha\gamma}\right]$$

where the terms inside [..] is equal to

$$\frac{(1-\gamma+\alpha\gamma)(1-\kappa)(\kappa-\alpha\gamma) + \sigma_2\,(1-\kappa)\,(\kappa(1-\gamma(1-\alpha)) - \alpha\gamma)}{(1-\kappa)\,(\kappa-\alpha\gamma)\,(\kappa(1-\gamma(1-\alpha)) - \alpha\gamma)}$$

i.e.

$$\frac{(1-\gamma+\alpha\gamma)(\kappa-\alpha\gamma) + \sigma_2\,(\kappa(1-\gamma(1-\alpha)) - \alpha\gamma)}{(\kappa-\alpha\gamma)\,(\kappa(1-\gamma(1-\alpha)) - \alpha\gamma)}$$

which is positive for all $\kappa \in [\underline{\kappa}, 1]$.

Combining (55) with (54) yields (20). ■